

2- C^* -Categories with non-simple units

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Abstract

We study the general structure of 2- C^* -categories closed under conjugation, projections and direct sums. We do not assume units to be simple, i.e. for ι_A the 1-unit corresponding to an object A , the space $\text{Hom}(\iota_A, \iota_A)$ is a commutative unital C^* -algebra. We show that 2-arrows can be viewed as continuous sections in Hilbert bundles and describe the behaviour of the fibres with respect to the categorical structure. We give an example of a 2- C^* -Category giving rise to bundles of finite Hopf-algebras in duality. We make some remarks concerning Frobenius algebras and Q -systems in the general context of tensor C^* -categories with non-simple units.

Introduction

Tensor categories arise in many mathematical contexts, such as quantum field theory, topological field theory, the theory of topological invariants for knots and links, quantum groups. The various examples may be characterised by more specific properties such as the notions of duality or of spherical (resp. braided, symmetric, modular) category. These have in common a “tensor” structure, i.e. a \otimes product defined on objects and on arrows and a particular object ι in the category, the unit with respect to this tensor product. Tensor C^* -categories are naturally related to the theory of operator algebras, and are at the heart of the structure of super-selection theory in algebraic quantum field theory and duality for compact groups (see [5], [6]).

A 2-category may be viewed as a generalisation of the notion of tensor category. We recover the notion of a tensor category as the particular case of a 2-category with only one object, where we can regard 1-arrows as the objects of the tensor category and 2-arrows as the arrows in the tensor category. The horizontal composition gives the tensor structure. 2- C^* -categories describe much of the structure in the theory of subfactors of von Neumann algebras. Many tools used in the past years which have proved to be fundamental in order to

characterise subfactors can be described in the context of 2- C^* -categories. Furthermore the language of categories makes easier the link with other fields of mathematics.

In a 2-category for each object A we have a particular 1-arrow $A \xleftarrow{\iota_A} A$, a unit with respect to the \otimes composition. In the tensor category case this corresponds to the unit object. In most of the literature units are supposed to be simple, i.e. $\text{End}(\iota_A) \equiv \text{Hom}(\iota_A, \iota_A) = \mathbb{K}1_{\iota_A}$, the spaces of 2-arrows connecting the unit to itself is a field. In this work we drop this hypothesis. More precisely, our object of study will be 2- C^* -categories closed under conjugation, projections and direct sums. The axioms for the \otimes , \circ products imply that the spaces $\text{End}(\iota_A)$ are commutative C^* -algebras, i.e. $\text{End}(\iota_A) \cong C(\Omega_A)$, the continuous functions over a compact Hausdorff space Ω_A . We show that for each couple $B \xleftarrow{\rho} A$, $B \xleftarrow{\sigma} A$ the space of 2-arrows $\text{Hom}(\rho, \sigma)$ connecting ρ to σ may be furnished with the structure of a Hilbert (or more generally a Banach) bundle over the topological space Ω_A , derived by the conjugation relations and depending on the choice of solutions up to continuous linear isomorphisms which respect the fibre structure (Proposition 2.2). This generalises the well known situation of simple units, where for each pair of 1-arrows ρ, σ the space of 2-arrows $\text{Hom}(\rho, \sigma)$ is finite dimensional. We show that each element of $\text{Hom}(\rho, \sigma)$ can be envisaged as a continuous section in this Banach bundle. We study the relation of this bundle structure with respect to the categorical structure, in particular how the fibres behave under \otimes and \circ composition, conjugation and $*$ involution. Furthermore, $\text{End}(\rho)$ has the structure of a C^* -algebra bundle, and this implies that even for the case $\rho \neq \sigma$ it is possible to give $\text{Hom}(\rho, \sigma)$ a Banach bundle structure such that the norm of each fibre satisfies the C^* -property. This fact implies, for example, that in the particular case of a $\text{End}(\iota)$ -linear tensor C^* -category one can describe the situation as that of a “bundle of tensor categories with simple units” (Proposition 2.32). We discuss the existence of a class of “standard” solutions to the conjugation equation, a generalisation of a notion appearing naturally in the simple units case, stable under composition of arrows, direct sums and projections.

The hypothesis of a non-simple unit has been considered by some authors in related contexts:

[1] and [2] deal with the categorical structure arising from the extension of a C^* -algebra with non trivial centre by a Hilbert C^* -system;

[8] deals with the existence of minimal conditional expectations for the inclusion of two von Neumann algebras with discrete centres;

[21] deals with the structure of depth two inclusions of finite dimensional C^* -algebras.

[11] studies a notion of Jones index and its relation to conjugation in the context of the 2- C^* -category of Hilbert C^* -bimodules;

[24] deals with crossed products of C^* -algebras with nontrivial centres by endomorphism, C^* -algebra bundles, group bundles.

Our work lives in a more abstract setting, and it may be viewed as a devel-

opment of some of the results of [16] for the case $\text{End}(\iota) \not\cong \mathbb{C}$.

The work is organised as follows:

- In the first part we recall the basic definitions and state some basic results concerning the structure of $2\text{-}C^*$ -categories.
- In the second part we develop further the structure of $2\text{-}C^*$ -categories. In particular the fibre structure appears.
- In the third part we define and study the class of “standard solutions “ to the conjugate equations, giving some conditions for their existence.
- In the fourth part we give an example of a 2-category generated by two conjugate 1-arrows $\rho, \bar{\rho}$ giving rise to two continuous bundles of finite dimensional Hopf-algebras in duality, a result analogous to the case of an irreducible depth two subfactor.
- In the fifth part we show, among other things, that in the tensor C^* -category context every Frobenius algebra is equivalent to a Q -system (Corollary 5.18). This fact strengthens a result contained in [18] dealing with the equivalence of Frobenius algebras and pairs of conjugate 1-arrows in a 2-category, as it implies that in the C^* - case some of the hypothesis assumed to hold are automatically verified (Proposition 5.20).

1 Preliminaries

We recall here briefly some basic concepts from category theory (the classical reference is [17], see also the introduction of [13] for a quick review). We will be interested only in small categories and we will not need more general definitions which avoid the notion of “set” from the beginning.

DEFINITION 1.1 *A category \mathcal{C} consists of a set of objects \mathcal{C}_0 and a set of arrows \mathcal{C}_1 together with the following structure:*

- a source map: $\mathcal{C}_1 \rightarrow \mathcal{C}_0$ assigning an object $s(f)$ to each arrow $f \in \mathcal{C}_1$,
- a target map: $\mathcal{C}_1 \rightarrow \mathcal{C}_0$ assigning an object $t(f)$ to each arrow $f \in \mathcal{C}_1$,
- an identity 1 map : $\mathcal{C}_0 \rightarrow \mathcal{C}_1$ assigning to each object a an arrow 1_a with $s(1_a) = t(1_a) = a$,
- a composition map $\circ : \mathcal{C}_1 \times \mathcal{C}_1 \rightarrow \mathcal{C}_1$ assigning to each pair of arrows f, g s.t. $s(g) = t(f)$ a third arrow $g \circ f$ with $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$.
- The composition \circ is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$ whenever these compositions make sense.
- the identity map satisfies $f \circ 1_a = f$, $\forall f$ s.t. $s(f) = a$ and $1_a \circ g = g$, $\forall g$ s.t. $t(g) = a$.

REMARK 1.2 An alternative way of defining a category is that of a set \mathcal{C} together with a set of pairs (f, g) of elements in \mathcal{C} , which are said to be composable, and a composition map $\circ : (f, g) \mapsto f \circ g$ assigning to each composable couple an element in \mathcal{C} . This map is supposed to be strictly associative. Elements u such that $f \circ u = f$ and $u \circ g = g$ for any f, g such that the compositions are defined are called units. For each $f \in \mathcal{C}$ there exist right and left units. Thus in this approach one deals only with arrows and the objects are identified with their units.

DEFINITION 1.3 A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ from the category \mathcal{C} to the category \mathcal{C}' consists of a map sending objects to objects and arrows to arrows such that:

- for any object $A \in \mathcal{C}_0$ we have $F(1_A) = 1_{F(A)}$
- for any arrow $f \in \mathcal{C}_1$ we have $s(F(f)) = F(s(f))$ and $t(F(f)) = F(t(f))$.
- $F(g \circ f) = F(g) \circ F(f)$, if f and g are composable.

DEFINITION 1.4 Let $F, G : \mathcal{C} \rightarrow \mathcal{C}'$ be functors from the category \mathcal{C} to the category \mathcal{C}' . A natural transformation $\eta : F \rightarrow G$ is a family of arrows $\eta(A) : F(A) \rightarrow G(A)$, $A \in \mathcal{C}$ in \mathcal{C}' such that for any morphism $f : A \rightarrow B$ in \mathcal{C} the following holds:

$$G(f) \circ \eta(A) = \eta(B) \circ F(f).$$

DEFINITION 1.5 A strict 2-category \mathcal{B} consists of

- a set \mathcal{B}_0 , whose elements are called objects.
- a set \mathcal{B}_1 , whose elements are called 1-arrows. A source map $s : \mathcal{B}_1 \rightarrow \mathcal{B}_0$ and a target map $t : \mathcal{B}_1 \rightarrow \mathcal{B}_0$ assigning to each 1-arrow a source and a target object.
- an identity map $\iota : \mathcal{B}_0 \rightarrow \mathcal{B}_1$ assigning to each object A a 1-arrow ι_A such that $s(\iota_A) = t(\iota_A) = A$.
- a composition map $\otimes : \mathcal{B}_1 \times \mathcal{B}_1 \rightarrow \mathcal{B}_1$ making $(\mathcal{B}_0, \mathcal{B}_1)$ into a category, i.e. for any $\rho, \sigma \in \mathcal{B}_1$ such that $s(\sigma) = t(\rho)$, $s(\sigma \otimes \rho) = s(\rho)$ and $t(\sigma \otimes \rho) = t(\sigma)$ hold.
- a set \mathcal{B}_2 , whose elements are called 2-arrows. A source map (we use the same symbols as above) $s : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ and a target map $t : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ assigning to each 2-arrow $S \in \mathcal{B}_2$ source and target 1-arrows. The maps s and t satisfy $s(s(S)) = s(t(S))$ and $t(t(S)) = t(s(S))$ for any $S \in \mathcal{B}_2$.
- a composition map (the “vertical” composition) $\circ : \mathcal{B}_2 \times \mathcal{B}_2 \rightarrow \mathcal{B}_2$ defined for each pair of S, T for \mathcal{B}_2 such that $s(T) = t(S)$.
- a unit map $1 : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ assigning to each 1-arrow ρ a 2-arrow 1_ρ such that $s(1_\rho) = \rho = t(1_\rho)$.

- a composition map (the “horizontal” composition, for which we use the same symbol as above) $\otimes : \mathcal{B}_2 \times \mathcal{B}_2 \rightarrow \mathcal{B}_2$ defined for each pair of 2-arrows S, T such that $s(s(T)) = t(s(S))$ and $s(t(T)) = t(t(S))$.
- the following equation for elements of \mathcal{B}_2 holds, whenever the compositions make sense

$$(S \otimes T) \circ (S' \otimes T') = (S \circ S') \otimes (T \circ T').$$

- for each $A \in \mathcal{B}_0$ the following equations hold, whenever the composition with elements $S, T \in \mathcal{B}_2$ make sense

$$T \otimes 1_{\iota_A} = T, \quad 1_{\iota_A} \otimes S = S.$$

Note that for any pair of objects A, B , we have a category $HOM(A, B)$, with the set of 1-arrows with source A and target B as objects of $HOM(A, B)$ and the set of 2-arrows with the latter as source and target as arrows and \circ as composition and 1 as unit map. Note that each object A determines a unit 1-arrow ι_A , which in turn determines a unit 2-arrow 1_{ι_A} .

REMARK 1.6 Concerning notation, we will give precedence, when not specified otherwise, to the \otimes product over the \circ product, i.e. $S \otimes T \circ U$ has to be read $(S \otimes T) \circ U$ and not $S \otimes (T \circ U)$.

REMARK 1.7 We will often write $\text{End}(\rho)$ instead of $\text{Hom}(\rho, \rho)$.

LEMMA 1.8 *The set of 2-arrows $\text{End}(\iota_A)$ is a commutative monoid.*

Proof. In fact for any $w, z \in \text{End}(\iota_A)$ we have $w \otimes z = (1_{\iota_A} \circ w) \otimes (z \circ 1_{\iota_A}) = (1_{\iota_A} \otimes z) \circ (w \circ 1_{\iota_A}) = z \circ w = (z \otimes 1_{\iota_A}) \circ (1_{\iota_A} \otimes w) = (z \circ 1_{\iota_A}) \otimes (1_{\iota_A} \circ w) = z \otimes w$, thus \otimes and \circ in this case agree and define a commutative monoid with unit 1_{ι_A} . ■

REMARK 1.9 As for categories, we can describe 2-categories in terms of 2-arrows. One considers a set with two operations \otimes, \circ and units such that each operation gives the set the structure of a category. Furthermore one supposes that all \otimes -units are also \circ -units and that an associativity relation as above

$$(S \otimes T) \circ (S' \otimes T') = (S \circ S') \otimes (T \circ T')$$

holds for the two products.

A weak 2-category (the term bicategory is also used) is a 2-category as above, where the associativity and unit identities are replaced by natural isomorphisms satisfying pentagon and triangle axioms. As we will deal (almost) only with the strict case, will not state these relations explicitly.

A 2- C^* -category is a 2-category for which the following hold:

- for each pair of 1-arrows ρ, σ the space $\text{Hom}(\rho, \sigma)$ is a complex Banach space.
- there is an anti-linear involution $*$ acting on 2-arrows, i.e. $*$: $\text{Hom}(\rho, \sigma) \rightarrow \text{Hom}(\sigma, \rho)$, $S \mapsto S^*$.
- the Banach norm is sub-multiplicative (i.e. $\|T \circ S\| \leq \|T\| \|S\|$, when the composition is defined) and satisfies the C^* -condition $\|S^* \circ S\| = \|S\|^2$.
- for any 2-arrow $S \in \text{Hom}(\rho, \sigma)$, $S^* \circ S$ is a positive element in $\text{End}(\rho)$.

REMARK 1.10 The above axioms imply that for each 1-arrow ρ the space $\text{End}(\rho)$ is a unital C^* -algebra. For each unit 1-arrow ι_A the space $\text{End}(\iota_A)$ is a commutative unital C^* -algebra.

We assume our category to be closed under projections (or retractions). By this we mean the following: take a 1-arrow $B \xleftarrow{\rho} A$ and consider the space $\text{End}(\rho)$, which has also the structure of an algebra, as we have seen. Then for each projection $P \in \text{End}(\rho)$ there exists a corresponding sub-1-arrow $A \xleftarrow{\rho_P} B$ and an isometry $W \in \text{Hom}(\rho_P, \rho)$ such that $W^* \circ W = 1_{\rho_P}$ and $W \circ W^* = P$.

We assume our category is closed under direct sums. By this we mean that for each pair of 1-arrows $B \xleftarrow{\rho_1, \rho_2} A$ there exists a 1-arrow $B \xleftarrow{\rho} A$ and isometries $W_1 \in \text{Hom}(\rho_1, \rho)$, $W_2 \in \text{Hom}(\rho_2, \rho)$ such that $W_1 \circ W_1^* + W_2 \circ W_2^* = 1_\rho$ and $W_i^* \circ W_j = 1_{\rho_i} \delta_{i,j}$. Consistently with the previous definition, ρ_1, ρ_2 are sub 1-arrows of ρ . We will sometimes simply write $\rho_1 \oplus \rho_2$ for a direct sum. Analogously we may identify a projection $P \in \text{End}(\rho)$ with the unit 1_{ρ_P} of its corresponding sub-1-arrow. Thus if $W_1, W_2, \rho_1, \rho_2, \rho$ are as above and $T_1 \in \text{End}(\rho_1)$, $T_2 \in \text{End}(\rho_2)$, we will simply indicate by $T_1 \oplus T_2 \in \text{End}(\rho_1 \oplus \rho_2)$ the 2-arrow $W_1 \circ T_1 \circ W_1^* + W_2 \circ T_2 \circ W_2^*$.

We assume that the category is closed under conjugation, that is, for each 1-arrow ρ going from A to B there exists another 1-arrow $\bar{\rho}$ from B to A and two 2-arrows $R_\rho \in \text{Hom}(\iota_A, \bar{\rho}\rho)$ and $\bar{R}_\rho \in \text{Hom}(\iota_B, \rho\bar{\rho})$ satisfying the following relations:

$$\bar{R}_\rho^* \otimes 1_\rho \circ 1_\rho \otimes R_\rho = 1_\rho; \quad R_\rho^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{R}_\rho = 1_{\bar{\rho}}.$$

This property is symmetric, i.e. if $\bar{\rho}$ is a conjugate for ρ , then ρ is a conjugate for $\bar{\rho}$, as is easily seen by taking $R_{\bar{\rho}} := \bar{R}_\rho$, $\bar{R}_{\bar{\rho}} := R_\rho$ as solutions. R_ρ and \bar{R}_ρ are fixed up to a choice of an invertible element in $\text{End}(\rho)$, i.e. if R'_ρ and \bar{R}'_ρ is another solution, then there exists an invertible $A \in \text{End}(\rho)$ such that $R'_\rho = (1_{\bar{\rho}} \otimes A) \circ R_\rho$ and $\bar{R}'_\rho = (A^{-1*} \otimes 1_\rho) \circ \bar{R}_\rho$. In fact, simply take $A = (\bar{R}_\rho^* \otimes 1_\rho) \circ (1_\rho \otimes R'_\rho)$. The same holds for $\text{End}(\bar{\rho})$.

Conjugacy is determined up to isomorphism, i.e. given conjugate a 1-arrows ρ and $\bar{\rho}$ with solution R_ρ, \bar{R}_ρ , any other 1-arrow $\bar{\rho}'$ conjugate to ρ is isomorphic to $\bar{\rho}$. In fact, let R'_ρ, \bar{R}'_ρ be solutions for ρ and $\bar{\rho}'$, then $(1_{\bar{\rho}} \otimes \bar{R}'_\rho) \circ (R_\rho \otimes 1_{\bar{\rho}'}) \in \text{Hom}(\bar{\rho}', \bar{\rho})$ is invertible.

Given two pairs of conjugate 1-arrows $\rho_1, \bar{\rho}_1$ and $\rho_2, \bar{\rho}_2$ with solutions R_1, \bar{R}_1 and R_2, \bar{R}_2 respectively, one can check that their sum $R_1 \oplus R_2, \bar{R}_1 \oplus \bar{R}_2$ is a solution for the couple of conjugate 1-arrows $\rho_1 \oplus \rho_2$ and $\bar{\rho}_1 \oplus \bar{\rho}_2$. Analogously given two pairs $\rho, \bar{\rho}$ and $\sigma, \bar{\sigma}$ with solutions R_ρ, \bar{R}_ρ and R_σ, \bar{R}_σ respectively, such that the composition $\sigma \otimes \rho$ is defined, one can consider the product solution for $\sigma \otimes \rho, \bar{\sigma} \otimes \bar{\rho}$ defined as $R_{\sigma \otimes \rho} := (1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho) \circ R_\rho$, $\bar{R}_{\sigma \otimes \rho} := (1_\sigma \otimes \bar{R}_\rho \otimes 1_{\bar{\sigma}}) \circ \bar{R}_\sigma$.

The conjugate relations imply, among other things, Frobenius duality, i.e. the following isomorphisms: $\text{Hom}(\rho, \sigma \otimes \eta) \cong \text{Hom}(\rho \otimes \bar{\eta}, \sigma) \cong \text{Hom}(\bar{\sigma} \otimes \rho, \eta) \cong \text{Hom}(\bar{\eta} \otimes \bar{\sigma} \otimes \rho, \iota_A)$.

We recall the definition of the \bullet map introduced in [16].

DEFINITION 1.11 *Given two 1-arrows ρ, σ , their conjugates $\bar{\rho}, \bar{\sigma}$, and a choice of solutions to the conjugation equations $R_\rho, \bar{R}_\rho, R_\sigma, \bar{R}_\sigma$, we define the map $\bullet : \text{Hom}(\rho, \sigma) \rightarrow \text{Hom}(\bar{\rho}, \bar{\sigma})$ by*

$$S^\bullet := (1_{\bar{\sigma}} \otimes \bar{R}_\rho^*) \circ (1_{\bar{\sigma}} \otimes S^* \otimes 1_{\bar{\rho}}) \circ (R_\sigma \otimes 1_{\bar{\rho}}), \quad \forall S \in \text{Hom}(\rho, \sigma),$$

and $\bullet : \text{Hom}(\bar{\rho}, \bar{\sigma}) \rightarrow \text{Hom}(\rho, \sigma)$ by

$$T^\bullet := (1_\sigma \otimes \bar{R}_\rho^*) \circ (1_\sigma \otimes T^* \otimes 1_\rho) \circ (R_{\bar{\sigma}} \otimes 1_\rho), \quad \forall T \in \text{Hom}(\bar{\rho}, \bar{\sigma}).$$

It is an anti-linear isomorphism, and its square is the identity. When $\rho = \sigma$ it is an algebraic anti-linear isomorphism and it satisfies $1_\rho^\bullet = 1_{\bar{\rho}}$. Notice that \bullet depends on the choice of the solution R, \bar{R} , and that in general it does not commute with the $*$ operation.

$R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are positive elements of the commutative C^* -algebras $\text{End}(\iota_A)$ and $\text{End}(\iota_B)$, respectively, so they can be thought of as positive functions in $C(\Omega_A)$ and $C(\Omega_B)$, Ω_A and Ω_B the spectra of the two commutative algebras.

But we can say more, the functions $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are strictly positive on their supports. Thus each 1-arrow ρ defines a projection in $C(\Omega_A)$, namely the projection on the (clopen) support of $R^* \circ R$, and analogously for the support of $\bar{R}^* \circ \bar{R}$ in $C(\Omega_B)$. These projections do not depend on the choice of the solutions of the conjugate equations R and \bar{R} . In particular, if Ω_A and Ω_B are connected, then $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are positive invertible functions.

These assertions are consequences of the following lemmas and propositions, most of which have been taken from [20] and [16].

LEMMA 1.12 *Let $w \in \text{End}(\iota_A)$, then the following conditions are equivalent:*

- a) $1_\rho \otimes w = 0$,
- b) $R^* \circ R \circ w = 0$.

Similarly if $z \in \text{End}(\iota_B)$ the following conditions are equivalent:

- a') $z \otimes 1_\rho = 0$,
- b') $z \circ \bar{R}^* \circ \bar{R} = 0$.

Proof. The implication $a) \Rightarrow b)$ is obvious. Suppose, without loss of generality, that w is positive. Then $R^* \circ R \circ w = R^* \otimes w^{\frac{1}{2}} \circ R \otimes w^{\frac{1}{2}} = 0$, which implies $R \otimes w^{\frac{1}{2}} = 0$ by the C^* -property of the norm. But then $1_\rho \otimes w^{\frac{1}{2}} = \bar{R}^* \otimes 1_\rho \circ 1_\rho \otimes R \otimes w^{\frac{1}{2}} = 0$, thus $1_\rho \otimes w = 1_\rho \otimes w^{\frac{1}{2}} \otimes w^{\frac{1}{2}} = 0$. The proof of $a') \Leftrightarrow b')$ is analogous. \blacksquare

The maps $\text{End}(\iota_A) \ni w \mapsto 1_\rho \otimes w \in Z(\text{End}(\rho))$ (the centre of $\text{End}(\rho)$) and $\text{End}(\iota_B) \ni z \mapsto z \otimes 1_\rho \in Z(\text{End}(\rho))$ are C^* homomorphisms into $Z(\text{End}(\rho))$. Denote by $S_l(\rho)$ and $S_r(\rho)$ the closed subspaces of Ω_A and Ω_B corresponding to the kernels of these maps.

LEMMA 1.13 $S_l(\rho)$ and $S_r(\rho)$ are the supports of $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ respectively.

Proof. Suppose $u \subset \Omega_A$ is an open subset such that $(R^* \circ R)|_u = 0$, then for any $w \in (\iota_A, \iota_A)$ with support in u we have $R^* \circ R \circ w = 0$. But this implies $1_\rho \otimes w = 0$ by Lemma 1.12, thus $u \cap S_l(\rho) = \emptyset$ and $S_l(\rho) \subset \text{supp}(R^* \circ R)$.

If $\omega \notin S_l(\rho)$ then since $S_l(\rho)$ is closed we can find a $w \in \text{End}(\iota_A)$ such that $\omega(w) \neq 0$ and $\omega'(w) = 0 \ \forall \omega' \in S_l(\rho)$. Thus $1_\rho \otimes w = 0$, so by Lemma 1.12 $0 = \omega(R^* \circ R \circ w) = \omega(R^* \circ R)\omega(w)$ which implies $\omega(R^* \circ R) = 0$ and $\text{supp}(R^* \circ R) \subset S_l(\rho)$. The proof for $S_r(\rho)$ is analogous. \blacksquare

COROLLARY 1.14 The supports of $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ do not depend on the choice of R and \bar{R} .

COROLLARY 1.15 Let $E_{S_l(\rho)}$ and $E_{S_r(\rho)}$ denote the projections onto the supports of $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ respectively. Then the following equalities hold: $1_\rho = 1_\rho \otimes E_{S_l(\rho)} = E_{S_r(\rho)} \otimes 1_\rho$.

LEMMA 1.16 The following inequalities hold:

$$\begin{aligned} R \circ R^* &\leq (R^* \circ R) \otimes 1_{\bar{\rho}\rho} \\ R \circ R^* &\leq 1_{\bar{\rho}\rho} \otimes (R^* \circ R) \\ \bar{R} \circ \bar{R}^* &\leq (\bar{R}^* \circ \bar{R}) \otimes 1_{\rho\bar{\rho}} \\ \bar{R} \circ \bar{R}^* &\leq 1_{\rho\bar{\rho}} \otimes (\bar{R}^* \circ \bar{R}). \end{aligned}$$

Proof. Notice that $(R \circ R^*) \circ (R \circ R^*) = (R \circ R^*) \otimes 1_{\bar{\rho}\rho} \otimes (R^* \circ R) = (R \circ R^*) \otimes (R^* \circ R) \otimes 1_{\bar{\rho}\rho}$, where we are regarding $(R \circ R^*)$, $1_{\bar{\rho}\rho} \otimes (R^* \circ R)$, $(R^* \circ R) \otimes 1_{\bar{\rho}\rho}$ as positive elements of the algebra $\text{End}(\rho)$. In particular $1_{\bar{\rho}\rho} \otimes (R^* \circ R)$ and $(R^* \circ R) \otimes 1_{\bar{\rho}\rho}$ are elements of the centre $Z(\text{End}(\bar{\rho}\rho))$. Analogous relations hold for $\bar{R} \circ \bar{R}^*$. Now, in general, if we have a positive element X in a C^* -algebra A such that $X^2 = XZ$, where Z is a positive element of the centre of A , we have $Z \geq X$. In fact, take a faithful representation (π, H) of the algebra A , take two generic vectors $\alpha \in H, \beta \in (X^{\frac{1}{2}}H)^\perp$. Then $(X^{\frac{1}{2}}\alpha + \beta, X(X^{\frac{1}{2}}\alpha + \beta)) = (\alpha, X^2\alpha)$ and $(X^{\frac{1}{2}}\alpha + \beta, Z(X^{\frac{1}{2}}\alpha + \beta)) = (\alpha, X^2\alpha) + (\beta, Z\beta)$. Thus $Z \geq X$. \blacksquare

PROPOSITION 1.17 *For each positive $X \in \text{End}(\rho \otimes \sigma)$ the following inequality holds:*

$$X \leq (\bar{R}^* \circ \bar{R}) \otimes 1_\rho \otimes (R^* \otimes 1_\sigma \circ (1_{\bar{\rho}} \otimes X) \circ R \otimes 1_\sigma).$$

Proof.

$$X = (1_\rho \otimes R^* \otimes 1_\sigma) \circ (\bar{R} \circ \bar{R}^*) \otimes X \circ (1_\rho \otimes R \otimes 1_\sigma) \leq$$

$$(1_\rho \otimes R^* \otimes 1_\sigma) \circ (\bar{R}^* \circ \bar{R}) \otimes 1_{\rho\bar{\rho}} \otimes X \circ (1_\rho \otimes R \otimes 1_\sigma)$$

$$= (\bar{R}^* \circ \bar{R}) \otimes 1_\rho \otimes (R^* \otimes 1_\sigma \circ (1_{\bar{\rho}} \otimes X) \circ R \otimes 1_\sigma),$$

where in the first line we have used the conjugation equations and in the second we have used the third inequality of the preceding lemma. \blacksquare

COROLLARY 1.18 *The following inequality holds:*

$$(\bar{R}^* \circ \bar{R}) \otimes 1_\rho \circ 1_\rho \otimes (R^* \circ R) \geq 1_\rho.$$

COROLLARY 1.19 *The following hold:*

- i) $1_\rho \otimes R^* \circ R \geq \frac{1}{\|\bar{R}\|^2} 1_\rho$; $\bar{R}^* \circ \bar{R} \otimes 1_\rho \geq \frac{1}{\|\bar{R}\|^2} 1_\rho$
- ii) $(R^* \circ R)|_{S_l(\rho)} \geq \frac{1}{\|\bar{R}\|^2}$; $(\bar{R}^* \circ \bar{R})|_{S_r(\rho)} \geq \frac{1}{\|\bar{R}\|^2}$
- iii) $S_l(\rho)$ and $S_r(\rho)$ are open and closed.

LEMMA 1.20 *Let ρ, σ be 1-arrows from A to B and $E_{S_l(\rho)}$ and $E_{S_l(\sigma)}$ the associated projections on $S_l(\rho)$, $S_l(\sigma)$ in $\text{Hom}(\iota_A, \iota_A)$. Suppose $E_{S_l(\rho)} E_{S_l(\sigma)} = 0$, then $\text{Hom}(\rho, \sigma) = 0$. An analogous assertion holds for the right supports $S_r(\rho), S_r(\sigma)$.*

Proof. If $T \in \text{Hom}(\rho, \sigma)$, then $T = 1_\sigma \circ T \circ 1_\rho = (1_\sigma \otimes E_{S_l(\sigma)}) \circ T \circ (1_\rho \otimes E_{S_l(\rho)}) = 1_\sigma \circ (T \otimes (E_{S_l(\sigma)} E_{S_l(\rho)})) \circ 1_\rho = 0$. \blacksquare

We will meet a refinement of this lemma in section 2.

We introduce now a construction which will be useful in the sequel. Suppose we have chosen for each object A of our 2- C^* -category \mathcal{A} a set of projections $\{1_{\iota_{A_i}}\}$ in the associated commutative C^* -algebra $\text{End}(\iota_A)$ such that $\sum_i 1_{\iota_{A_i}} = 1_{\iota_A}$ (i.e. the set is complete) and $1_{\iota_{A_i}} \otimes 1_{\iota_{A_j}} = \delta_{i,j} 1_{\iota_{A_i}}$ (i.e. the projections are orthogonal). To each projection $1_{\iota_{A_i}}$ there will correspond a 1-arrow, which we will call ι_{A_i} . We would like to think of these 1-arrows as units corresponding to objects. In other words, we would like to “decompose”, in some sense, each object into sub-objects corresponding to our original choice of sets of projections. 1-arrows and 2-arrows should be decomposed accordingly. We must show that this can be done in a consistent manner. We define a new 2- C^* -category \mathcal{B} the following way (we will use the “only 2-arrows approach” mentioned above):

- define as 2-arrows of the new category \mathcal{B} all the elements of the form

$$\{1_{\iota_{B_j}} \otimes S \otimes 1_{\iota_{A_i}}, \forall 1_{\iota_{B_j}}, 1_{\iota_{A_i}}, B \xleftarrow{\rho} A, B \xleftarrow{\sigma} A, S \in \text{Hom}(\rho, \sigma), A, B \in \mathcal{A}\}$$

with the same \otimes and \circ operations of the original category.

- we set each projection $1_{\iota_{A_i}}$ to be a \otimes -unit (thus, also a \circ -unit),
- we define the \circ -units to be the set $\{1_{\iota_{B_j}} \otimes 1_{\rho} \otimes 1_{\iota_{A_i}}, \forall 1_{\iota_{A_i}}, 1_{\iota_{B_j}}, A \xleftarrow{\rho} B\}$

The units satisfy the necessary properties by definition and compatibility between the \otimes and the \circ products descends from the original one. The new 2-category is still closed under conjugation. In fact, let $B \xleftarrow{\rho} A$ and $A \xleftarrow{\bar{\rho}} B$ be two conjugate 1-arrows in the original category. Then each $1_{\iota_{B_j}} \otimes 1_{\rho} \otimes 1_{\iota_{A_i}}$ has as conjugate $1_{\iota_{A_i}} \otimes 1_{\bar{\rho}} \otimes 1_{\iota_{B_j}}$ with conjugate solutions $R_{i,j} := (1_{\bar{\rho}} \otimes 1_{\iota_{B_j}} \otimes 1_{\rho}) \circ R \circ 1_{\iota_{A_i}}$ and $\bar{R}_{i,j} := (1_{\rho} \otimes 1_{\iota_{A_i}} \otimes 1_{\bar{\rho}}) \circ \bar{R} \circ 1_{\iota_{B_j}}$.

REMARK 1.21 The above construction is not an inclusion of the $2\text{-}C^*$ -category \mathcal{A} in \mathcal{B} by in the proper sense. In fact, each unit 1_{ι_A} is not sent to one but to a set of units $1_{\iota_{A_i}}$. The original $2\text{-}C^*$ -category \mathcal{A} is easily recovered from the new one by considering linear combinations of the elements of \mathcal{B} (the fact that linear combinations of 2-arrows in \mathcal{B} span all of the 2-arrows in \mathcal{A} is a consequence of Lemma 1.20).

REMARK 1.22 Obviously this construction depends on the choice of the sets of projections $\{1_{\iota_{A_i}}\}$. If we suppose the topological spaces Ω_A corresponding to each object A to be a finite union of connected components $\{\Omega_{A_i}\}$ then the central projections associated to each component would be a natural choice. Notice that in this case in the new category \mathcal{B} the objects A_i would have connected spectrum, thus for any $B_j \xleftarrow{\rho} A_i$ the elements $R_{\rho}^* \circ R_{\rho}$ and $\bar{R}_{\rho}^* \circ \bar{R}_{\rho}$ would be invertible as a consequence of Corollary 1.19.

2 Bimodules and bundles.

In this section we will pursue a parallel path with respect to the case of simple units. We will show that Banach, Hilbert and C^* -algebra bundles appear as a generalisation of finite dimensional spaces. Several analogous results are obtained, such as a finite upper bound on the dimension of the fibres, given by the non scalar analogue of the dimension function. We investigate the behaviour of these fibres with respect to the categorical structure: the \circ composition, the $*$ involution and conjugation naturally preserve the fibres and are straightforward to describe. The behaviour under the \otimes composition is less friendly and for this purpose we introduce a further hypothesis. Our assumption is not the most general a priori, but general enough to comprehend most (if not all) of the known examples (e.g. $\text{End}(\iota)$ -linear tensor categories, such as braided categories, are

included). Furthermore it does not seem easy to produce a counterexample, a question we leave open for the future.

We begin by noticing that the spaces $\text{Hom}(\rho, \sigma)$ have a structure of $\text{End}(\iota_B)$ – $\text{End}(\iota_A)$ Hilbert bimodule given by the conjugation relations and the \otimes product. In fact, given $z \in \text{End}(\iota_B)$ and $w \in \text{End}(\iota_A)$ and $T \in \text{Hom}(\rho, \sigma)$, we can consider the tensor products $z \otimes T$ and $T \otimes w$ both still in $\text{Hom}(\rho, \sigma)$. Given $T, S \in \text{Hom}(\rho, \sigma)$ we define the right $\text{End}(\iota_A)$ -valued product $\langle T, S \rangle_{\text{End}(\iota_A)}^{(\rho, \sigma)}$ by $R^* \circ (1_{\bar{\rho}} \otimes (S^* \circ T)) \circ R$. The same way we define the left $\text{End}(\iota_B)$ -valued product $\langle T, S \rangle_{\text{End}(\iota_B)}^{(\rho, \sigma)}$ by $\bar{R}^* \circ ((S^* \circ T) \otimes 1_{\bar{\rho}}) \circ \bar{R}$. Both of these products are non-degenerate.

We recall the definition of a Banach bundle (the terminology “continuous field of Banach spaces” is also used in the literature, cf. e.g., [4]).

DEFINITION 2.1 *Let Ω be a compact Hausdorff topological space. A Banach bundle E over Ω is a family of Banach spaces $\{E_\omega, \|\cdot\|^\omega, \omega \in \Omega\}$ with a set $\Gamma \subset \prod_{\omega \in \Omega} E_\omega$ such that:*

- i) Γ is a linear subspace of $\prod_{\omega \in \Omega} E_\omega$*
- ii) $\forall \omega \in \Omega$ $\{S|_\omega, S \in \Gamma\}$ is dense in E_ω*
- iii) $\forall S \in \Gamma$ the norm function $\omega \rightarrow \|S|_\omega\|^\omega$ is a continuous function on Ω*
- iv) Let $X \in \prod_{\omega \in \Omega} E_\omega$. If $\forall \omega \in \Omega$ and $\forall \epsilon > 0 \exists S \in \Gamma$ such that $\|X|_\omega - S|_\omega\|^\omega < \epsilon$ in a neighbourhood of ω , then $X \in \Gamma$.*

The elements of $\prod_{\omega \in \Omega_A} E_\omega$ are called sections and those of Γ continuous sections. Analogously if the spaces E_ω have the structure of Hilbert spaces and the norm is given by the inner product, we will talk about Hilbert bundles. If the E_ω have the structure of C^* -algebras, and the space of continuous sections Γ is closed under multiplication and the $*$ operation, we will talk about a C^* -algebra bundle. In a Banach bundle the fibre space might vary according to the base point. So it is a more general situation than that of a locally trivial bundle. The choice of a set $\Gamma \subset \prod_{\omega} E_\omega$ as the space of continuous sections is part of the initial data, as in general we have no local charts, with the implicit notion of continuity given by them.

PROPOSITION 2.2 *Given two 1-arrows ρ, σ from objects A to B and a choice of the conjugation equations R_ρ, \bar{R}_ρ for ρ , $\text{Hom}(\rho, \sigma)$ has the structure of a Hilbert bundle.*

Proof. We evaluate the product $\langle S, T \rangle_{\text{End}(\iota_A)}^{(\rho, \sigma)}$ on each $\omega \in \Omega_A$ for any $S, T \in \text{Hom}(\rho, \sigma)$. The procedure of the GNS construction gives us for each point ω a Hilbert space, which we shall denote by $\text{Hom}(\rho, \sigma)_\omega$. We take $\prod_{\omega \in \Omega_A} \text{Hom}(\rho, \sigma)_\omega$ as fibre bundle and the image of $\text{Hom}(\rho, \sigma)$ (which we will still denote by $\text{Hom}(\rho, \sigma)$) as the module of continuous sections.

Note that for $S \in \text{Hom}(\rho, \sigma)$ the topology given by $\sup_{\omega \in \Omega} \|S_\omega\|^\omega$ is equiv-

alent to the original one. In fact, on the one hand we have

$$\begin{aligned} \sup_{\omega \in \Omega_A} \|S|_{\omega}\|^{\omega} &= \|\langle S, S \rangle_{\text{End}(\iota_A)}^{\frac{1}{2}}\| = \|\langle S, S \rangle_{\text{End}(\iota_A)}\|^{\frac{1}{2}} \\ &\leq \|S^* S\|^{\frac{1}{2}} \|\langle 1_{\rho}, 1_{\rho} \rangle_{\text{End}(\iota_A)}\|^{\frac{1}{2}} = \|S^* S\|^{\frac{1}{2}} \|R_{\rho}^* \circ R_{\rho}\|^{\frac{1}{2}} = \|S\| \|R_{\rho}\| \end{aligned}$$

and on the other hand we have

$$\begin{aligned} \|S\| &= \|S^* \circ S\|^{\frac{1}{2}} \leq \|(\bar{R}_{\rho}^* \circ \bar{R}_{\rho}) \otimes 1_{\rho} \otimes \langle S, S \rangle_{\text{End}(\iota_A)}\|^{\frac{1}{2}} \\ &\leq \|(\bar{R}_{\rho}^* \circ \bar{R}_{\rho})\|^{\frac{1}{2}} \|\langle S, S \rangle_{\text{End}(\iota_A)}\|^{\frac{1}{2}} = \|\bar{R}_{\rho}^* \circ \bar{R}_{\rho}\|^{\frac{1}{2}} \|\langle S, S \rangle_{\text{End}(\iota_A)} S\|^{\frac{1}{2}} \\ &= \|\bar{R}_{\rho}\| (\sup_{\omega \in \Omega_A} \|S_{\omega}\|^{\omega}) = \|\bar{R}_{\rho}\| \|S_{\omega}\|, \end{aligned}$$

where we have used the expression of the definition of the inner product $\langle \cdot, \cdot \rangle_{\text{End}(\iota_A)}$, the monotonicity of the square root function and the inequality for $S^* S$ given by Proposition 1.17. So $\text{Hom}(\rho, \sigma)$ is closed even as a subspace of the Banach bundle. The first three conditions either very easy to prove. We prove only the last one. Suppose we have $X \in \prod_{\omega \in \Omega_A} \text{Hom}(\rho, \sigma)_{\omega}$ satisfying condition *iv*). Then, as Ω_A is compact, $\forall \epsilon > 0$ we can choose a finite family of elements $S^{\alpha} \in \text{Hom}(\rho, \sigma)$ and a corresponding finite open covering $\{U_{\alpha}\}$ of Ω_A such that $\|X - S^{\alpha}\|_{U_{\alpha}}^{\omega} \leq \epsilon$. Take a partition of unity f_{α} subordinate to the open covering. Then $\|X - \sum_{\alpha} f_{\alpha} S^{\alpha}\|^{\omega} \leq \epsilon$ because of convexity of the norm. Thus $X \in \overline{\text{Hom}(\rho, \sigma)} = \text{Hom}(\rho, \sigma)$. \blacksquare

This gives the $\text{End}(\iota_B) - \text{End}(\iota_A)$ bimodule $\text{Hom}(\rho, \sigma)$ the structure of a Hilbert bundle over the compact topological space Ω_A . So we can think of each element T as a continuous section $T|_{\omega}$ in this Hilbert bundle. The right action of $\text{End}(\iota_A)$ is given by multiplying functions in $C(\Omega_A)$.

Having shown that each $\text{Hom}(\rho, \sigma)$ has a Hilbert bundle structure, we would like to show, as already mentioned, its behaviour in the context of the whole category. In fact $\text{Hom}(\rho, \sigma)$ is not only a bimodule: its elements can be regarded as operators between other spaces by \circ composition of 2-arrows. For example the elements of $\text{Hom}(\rho, \sigma)$ can be regarded as operators from $\text{Hom}(\eta, \rho)$ to $\text{Hom}(\eta, \sigma)$. Suppose we have chosen solutions to the conjugation equations for ρ and for η , with the respective induced Hilbert bundle structures. We have the following

PROPOSITION 2.3 *If $T, T' \in \text{Hom}(\rho, \sigma)$ such that $T|_{\omega} = T'|_{\omega}$, $P, P' \in \text{Hom}(\eta, \rho)$ such that $P|_{\omega} = P'|_{\omega}$ then $(T \circ P)|_{\omega} = (T' \circ P)|_{\omega} = (T \circ P')|_{\omega} = (T' \circ P')|_{\omega}$.*

Proof. $T_{\omega} = T'_{\omega}$ means $\langle T, S \rangle_{\text{End}(\iota_A)}^{(\rho, \sigma)}|_{\omega} = \langle T', S \rangle_{\text{End}(\iota_A)}^{(\rho, \sigma)}|_{\omega} \forall S \in \text{Hom}(\rho, \sigma)$. Analogous relations hold for $P, P' \in \text{Hom}(\eta, \rho)$. We have

$$\begin{aligned} \langle T \circ P, Q \rangle_{\text{End}(\iota_A)}^{(\eta, \sigma)}|_{\omega} &= \langle P, T^* \circ Q \rangle_{\text{End}(\iota_A)}^{(\eta, \sigma)}|_{\omega} = \langle P', T^* \circ Q \rangle_{\text{End}(\iota_A)}^{(\eta, \sigma)}|_{\omega} \\ &= \langle T \circ P', Q \rangle_{\text{End}(\iota_A)}^{(\eta, \sigma)}|_{\omega}, \quad \forall Q \in \text{Hom}(\eta, \rho). \end{aligned}$$

Thus $(T \circ P)|_\omega = (T \circ P')|_\omega$, and in the same way we have $(T' \circ P)|_\omega = (T' \circ P')|_\omega$.

Also we have

$$\begin{aligned} \langle T \circ P, Q \rangle_{\text{End}(\iota_A)}^{(\eta, \sigma)}|_\omega &= \langle T, Q \circ P^{*\bullet\bullet\bullet*} \rangle_{\text{End}(\iota_A)}^{(\rho, \sigma)}|_\omega = \langle T', Q \circ P^{*\bullet\bullet\bullet*} \rangle_{\text{End}(\iota_A)}^{(\rho, \sigma)}|_\omega \\ &= \langle T' \circ P, Q \rangle_{\text{End}(\iota_A)}^{(\eta, \sigma)}|_\omega, \quad \forall Q \in \text{Hom}(\eta, \rho), \end{aligned}$$

thus $(T \circ P)|_\omega = (T' \circ P)|_\omega$. ■

The preceding proposition implies the following

COROLLARY 2.4 *For each $\omega \in \Omega_A$ the set*

$$I_\omega := \{S \in \text{End}(\rho) \text{ s.t. } \langle S, S \rangle_{\text{End}(\iota_A)}|_\omega = 0\}$$

is a closed two-sided ideal of $\text{End}(\rho)$.

COROLLARY 2.5 *Choose solutions R_ρ, \bar{R}_ρ and R_σ, \bar{R}_σ for ρ and σ , respectively. For $T, T' \in \text{Hom}(\rho, \sigma)$ if $T|_\omega = T'|_\omega$ then $T|_\omega^* = T'|_\omega^*$.*

Proof. By the previous proposition if $T|_\omega = T'|_\omega$ (i.e. $(T - T')|_\omega = 0$) then $((T - T') \circ (T - T')^*)|_\omega = 0$, which implies $\langle ((T - T') \circ (T - T')^*, 1_\sigma) \rangle_{\text{End}(\iota_A)}^{(\sigma, \rho)}|_\omega = \langle (T - T')^*, (T - T')^* \rangle_{\text{End}(\iota_A)}^{(\sigma, \rho)}|_\omega = 0$, i.e. $T|_\omega^* = T'|_\omega^*$. ■

Thus the \circ and $*$ operations preserve the fibre structure.

REMARK 2.6 $\text{End}(\rho)/I_\omega$ is the pre-Hilbert space that gives rise to the fibre-Hilbert space $\text{End}(\rho)_\omega$ when completed with respect to the pre-scalar product norm. For each ω we can pursue the whole GNS construction and obtain a C^* -algebra $\pi_\omega(\text{End}(\rho))$ acting on this Hilbert space. The preceding corollary shows that $\pi_\omega(\text{End}(\rho))$ is the completion of the same pre-Hilbert space $\text{End}(\rho)/I_\omega$ with respect to the C^* -norm given by the GNS construction.

We have the following

PROPOSITION 2.7 *$\text{End}(\rho)$ has the structure of a C^* -algebra bundle*

Proof. Proceed as in the beginning of Proposition 2.2 and for each ω consider the GNS construction. We must show the continuity of this C^* -norm with respect to the base point ω . For $A \in \text{End}(\rho)/I_\omega$ we define

$$\|A\|_\omega^{C^*} := \sup_{\tilde{y} \in \text{End}(\rho)/I_\omega, \|\tilde{y}\|^{Hilbert} \leq 1} \|A\tilde{y}\|_\omega^{Hilbert},$$

where by $\|A\tilde{y}\|_\omega^{Hilbert}$ we mean $\langle Ay, Ay \rangle_\omega^{\frac{1}{2}}$, for any $y \in \text{End}(\rho)$ such that $y|_\omega = \tilde{y}$. As this norm is defined as a sup over continuous functions, it is a priori only lower semicontinuous. As I_ω is a two-sided C^* -ideal, there is another candidate C^* -norm, namely $\|A\|_\omega^{C^*2} := \inf_{y \in I_\omega} \|A - y\|$, the C^* -norm of the quotient C^* -algebra $\text{End}(\rho)/I_\omega$. We show that these two norms are the same.

LEMMA 2.8 $\|\cdot\|_{\omega}^{C_2^*} = \|\cdot\|_{\omega}^{C_1^*}$.

Proof. Take an approximate unit u_λ for I_ω . Then

$$\begin{aligned} \inf_{y \in I_\omega} \|A - y\| &= \lim_{\lambda \rightarrow \infty} \|A(1 - u_\lambda)\| = \\ \lim_{\lambda \rightarrow \infty} \sup_{\phi \in \mathcal{S}\text{End}(\rho)} \phi((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} &= \\ \lim_{\lambda \rightarrow \infty} \sup_{\phi \in \mathcal{P}\text{End}(\rho)} \phi((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}}, \end{aligned}$$

where $\mathcal{S}\text{End}(\rho)$ and $\mathcal{P}\text{End}(\rho)$ are the states and the pure states, respectively, of $\text{End}(\rho)$.

Notice that as $\langle \cdot, \cdot \rangle$ is a non-degenerate $C(\Omega_A)$ -valued inner product, we can restrict ourselves evaluating the supremum on pure states of the algebra $\text{End}(\rho)$ dominated by states of the form $\langle y, y \rangle_{|\omega'}$ for some $\omega' \in \Omega_A$ and some $y \in \text{End}(\rho)$ such that $\langle y, y \rangle_{|\omega'} = 1$.

For each u_λ choose a sequence $\phi_n^\lambda \in \mathcal{P}\text{End}(\rho)$ such that $\lim_{n \rightarrow \infty} \phi_n^\lambda((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} = \|A(1 - u_\lambda)\|$. Then choose a diagonal sequence ϕ_λ such that $\lim_{\lambda \rightarrow \infty} \phi_\lambda((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} = \lim_{\lambda \rightarrow \infty} \|A(1 - u_\lambda)\|$. Let ϕ_0 be an accumulation point of this last sequence. Then $\lim_{\lambda \rightarrow \infty} \phi_0((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} = \lim_{\lambda \rightarrow \infty} \|A(1 - u_\lambda)\|$. Suppose that ϕ_0 is dominated by a state of the kind $\langle y, y \rangle_{|\omega'}$ for some $\omega' \neq \omega$. Then choosing a continuous function $g \in C(\Omega_A)$ such that $g(\omega') = 1$ and $g(\omega) = 0$ we have $\lim_{\lambda \rightarrow \infty} \phi_0((A(1 - u_\lambda))^* A(1 - u_\lambda)) = \lim_{\lambda \rightarrow \infty} \phi_0(g(A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}}$. But the last term is zero, as $g \in I_\omega$. So any accumulation point must be dominated by a state of the form $\langle y, y \rangle_{|\omega}$. But in ω each $u_\lambda = 0$, so $\phi_0((A(1 - u_\lambda))^* A(1 - u_\lambda)) = \phi_0(A^* A)$.

So we conclude

$$\begin{aligned} \|A\|_{\omega}^{C_2^*} &= \inf_{y \in I_\omega} \|A - y\| = \phi_0(A^* A)^{\frac{1}{2}} = \\ \sup_{\tilde{y} \in \text{End}(\rho)/I_\omega, \|\tilde{y}\|_{\omega}^{\text{Hilbert}}=1} \langle A\tilde{y}, A\tilde{y} \rangle_{|\omega}^{\frac{1}{2}} &= \|A\|_{\omega}^{C_1^*}. \end{aligned}$$

Now we must show that $\|\cdot\|_{\omega}^{C_2^*}$ is upper semicontinuous. Suppose $y^\omega \in I_\omega$ such that $\|A - y^\omega\| = \|A\|_{\omega}^{C_2^*} + \epsilon$. Then choose a neighbourhood $U_\omega \subset \Omega_A$ such that $\|y^\omega\|_{\omega'}^{\text{Hilbert}} \leq \epsilon \forall \omega' \in U_\omega$.

Then $\|A\|_{\omega'}^{C_2^*} = \inf_{y^{\omega'} \in I_{\omega'}} \|A - y^{\omega'}\| \leq \inf_{y^{\omega'} \in I_{\omega'}} (\|A - y^\omega\| + \|y^\omega - y^{\omega'}\|)$.

But $\|y^\omega - y^{\omega'}\| \leq \text{const} \|\langle (y^\omega - y^{\omega'}), (y^\omega - y^{\omega'}) \rangle\|^{\frac{1}{2}}$, where the constant is independent of $y^\omega, y^{\omega'}$ and is given by Lemma 1.17, and $\inf_{y^{\omega'} \in I_{\omega'}} \|\langle (y^\omega - y^{\omega'}), (y^\omega - y^{\omega'}) \rangle\|^{\frac{1}{2}} \leq \epsilon$. Thus $\|A\|_{\omega'}^{C_2^*} \leq \|A\|_{\omega}^{C_2^*} + \epsilon + \text{const}\epsilon$ for any $\omega' \in U_\omega$, which means that $\|\cdot\|_{\omega}^{C_2^*}$ is upper semicontinuous.

Having proven continuity of the C^* -norm, the rest of the proof follows as in Proposition 2.2. ■

REMARK 2.9 We have introduced two kinds of bundle structures for $\text{End}(\rho)$, the first one giving each fibre a scalar product norm, the second a C^* -norm. As the inner $\text{End}(\iota_A)$ and $\text{End}(\iota_B)$ products depend on the choice of solutions to the conjugation equation, the Hilbert bundle structure is defined only up to an isomorphism. In fact, consider $\text{Hom}(\rho, \sigma)$ with the bundle structure given by a solution R, \bar{R} for ρ and a second bundle structure given by a second solution $(1_{\bar{\rho}} \otimes A) \circ R, (A^{-1*} \otimes 1_{\bar{\rho}}) \circ \bar{R}$, where $A \in \text{End}(\rho)$ is an invertible. Then the map $S \mapsto S \circ A^{-1} \forall S \in \text{Hom}(\rho, \sigma)$ is a unitary map between the two Hilbert bundles structures.

Even for $\rho \neq \sigma$ we can define for $\text{Hom}(\rho, \sigma)$ a Banach bundle structure where the fibre norm satisfies a C^* -condition by defining $\|S|_{|\omega}\|_{\omega} := (\|(S^* \circ S)|_{|\omega}\|_{\omega}^{C^*})^{\frac{1}{2}}$. We will consider this structure by default, if not specified otherwise. As we shall see in the sequel, the fibres are finite dimensional, so the two types of norms are equivalent. The latter bundle structure endows each Banach space fibre with its unique C^* -norm.

We also have the following

PROPOSITION 2.10 *For each $\text{End}(\rho)$ and for each point ω of the base space Ω the associated C^* -algebra fibre space $\text{End}(\rho)_{\omega}$ is finite dimensional.*

Proof. This is essentially the same proof as in [16]. Consider a set $\{X_i|_{|\omega} \in \text{End}(\rho)_{\omega}\}$ of positive elements of norm one with $\sum_i X_i|_{|\omega} \leq 1_{\rho|_{|\omega}}$. We can think, without loss of generality, of each $X_i|_{|\omega}$ as the value in ω of a positive $X_i \in \text{End}(\rho)$ (if not, just take $(X_i^* \circ X_i)^{\frac{1}{2}}$ instead). By the inequality in Corollary 2.27 we have

$$X_i \leq ((\bar{R}^* \circ \bar{R}) \otimes 1_{\rho} \otimes (R^* \circ 1_{\bar{\rho}} \otimes X_i \circ R)) = (\bar{R}^* \circ \bar{R}) \otimes 1_{\rho} \otimes \langle X_i, 1_{\rho} \rangle_{\text{End}(\iota_A)}.$$

Notice that $((\bar{R}^* \circ \bar{R}) \otimes 1_{\rho} \otimes \langle X_i, 1_{\rho} \rangle_{\text{End}(\iota_A)})|_{|\omega}$ is simply $((\bar{R}^* \otimes \bar{R}) \otimes 1_{\rho})|_{|\omega}$ times the positive constant $(\langle X_i, 1_{\rho} \rangle_{\text{End}(\iota_A)})|_{|\omega}$. As the norm of each single $X_i|_{|\omega}$ is one, we have

$$1 = \|X_i|_{|\omega}\|_{\omega} \leq \|((\bar{R}^* \circ \bar{R}) \otimes 1_{\rho})|_{|\omega}\|_{\omega}^{\omega} (\langle X_i, 1_{\rho} \rangle_{\text{End}(\iota_A)})|_{|\omega}.$$

Summing over i we have

$$n = \sum_i \|X_i|_{|\omega}\|_{\omega} \leq \sum_i \|((\bar{R}^* \circ \bar{R}) \otimes 1_{\rho})|_{|\omega}\|_{\omega}^{\omega} (\langle X_i, 1_{\rho} \rangle_{\text{End}(\iota_A)})|_{|\omega}$$

$$\leq \|((\bar{R}^* \circ \bar{R}) \otimes 1_{\rho})|_{|\omega}\|_{\omega}^{\omega} (\langle 1_{\rho}, 1_{\rho} \rangle_{\text{End}(\iota_A)})|_{|\omega} = \|((\bar{R}^* \circ \bar{R}) \otimes 1_{\rho})|_{|\omega}\|_{\omega}^{\omega} (R^* \circ R)|_{|\omega},$$

which is finite. ■

REMARK 2.11 A posteriori we see that the fibre spaces $\text{End}(\rho)/I_{\omega}$ are finite dimensional, thus the completion in the Hilbert and the C^* -norm was superfluous. They are finite dimensional C^* -algebras.

COROLLARY 2.12 *For each $\text{Hom}(\rho, \sigma)$ and for each point of the base space Ω the associated Hilbert fibre space $\text{Hom}(\rho, \sigma)_\omega$ is finite dimensional.*

Proof. That $\text{Hom}(\sigma, \rho)_\omega \circ \text{Hom}(\rho, \sigma)_\omega$ is finite dimensional follows from the preceding proposition and remark by embedding it into $\text{End}(\rho)_\omega$. But the map $\text{Hom}(\rho, \sigma)_\omega \rightarrow \text{Hom}(\rho, \sigma)_\omega \circ \text{Hom}(\sigma, \rho)_\omega : S_\omega \mapsto S_\omega^* \circ S_\omega$, $S_\omega \in \text{Hom}(\rho, \sigma)_\omega$ is injective. Thus $\text{Hom}(\rho, \sigma)_\omega$ is finite dimensional as well. ■

So far we were able to obtain many results under fairly general assumptions and in a straightforward way. In particular, a picture analogous to the case of simple units appears, where finite dimensional spaces are replaced by Banach bundles with finite dimensional fibres and \circ composition of 2-arrow takes place fibrewise. As already mentioned, the behaviour under the \otimes composition is harder to describe, and in order to do so we introduce some additional hypothesis.

We begin with the following definition:

DEFINITION 2.13 *We call a 1-arrow $B \xleftarrow{\rho} A$ “centrally balanced” if the following holds:*

$$\text{End}(\iota_B) \otimes 1_\rho = 1_\rho \otimes \text{End}(\iota_A).$$

We don’t claim that 2-arrows of this form exhaust all of $Z(\text{End}(\rho))$, but simply that for each $z \in \text{End}(\iota_B)$ there exists $w \in \text{End}(\iota_A)$ s.t. $z \otimes 1_\rho = 1_\rho \otimes w$ (and vice-versa). Notice that this property is closed for \otimes products and sub-objects.

Now we make the following

ASSUMPTION 2.14 (BALANCED DECOMPOSITION) *We assume every 1-arrow ρ to be a direct sum $\oplus_i \rho_i$ of centrally balanced 1-arrows.*

We have the following

PROPOSITION 2.15 *Let $B \xleftarrow{\rho} A$ be a centrally balanced 1-arrow with $S_l(\rho) = \Omega_A$ and $S_r(\rho) = \Omega_B$. Then ρ establishes an isomorphism of the two algebras $\text{End}(\iota_A)$ and $\text{End}(\iota_B)$. The isomorphism, which we shall denote $\theta_\rho : \text{End}(\iota_A) \rightarrow \text{End}(\iota_B)$, is independent of the choice of solutions of the conjugation equation and is given by the expression $\theta_\rho(w) = \frac{\text{End}(\iota_B) \langle 1_\rho \otimes w, 1_\rho \rangle^{(\rho, \rho)}}{\text{End}(\iota_B) \langle 1_\rho, 1_\rho \rangle^{(\rho, \rho)}}$.*

Proof. The first sentence is just a restatement of Definition 2.13: the maps $s \mapsto s \otimes 1_\rho$ and $s \mapsto 1_\rho \otimes s$, $s \in \text{End}(\iota_B)$, are injective (this follows from Lemma 1.12 and the fact that $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are invertible, as $S_l(\rho) = \Omega_A$ and $S_r(\rho) = \Omega_B$). Definition 2.13 tells us that they have the same images in $Z(\text{End}(\rho))$.

In order to give the rest of the proof we first introduce a simple lemma:

LEMMA 2.16 *For ρ as above one has $w \otimes 1_{\bar{\rho}\rho} = 1_{\bar{\rho}\rho} \otimes w$ for all $w \in \text{End}(\iota_A)$.*

Proof. As there exists $w' \in \text{End}(\iota_A)$ s.t. $w \otimes 1_{\bar{\rho}\rho} = 1_{\bar{\rho}\rho} \otimes w'$, we only have to prove that they are the same. But $w \otimes 1_{\bar{\rho}\rho} = 1_{\bar{\rho}\rho} \otimes w'$ implies $w \otimes R = R \otimes w'$. And $w \otimes R = R \circ w = R \otimes w$ which implies $w' = w$, as we have shown that tensoring elements of $\text{End}(\iota_A)$ with R is an injective map when $R^* \circ R$ is invertible. \blacksquare

We show that $\theta_\rho(w) \otimes 1_\rho = 1_\rho \otimes w$. To do so we take the difference of the two elements and evaluate the product $\langle (1_\rho \otimes w - \theta_\rho(w) \otimes 1_\rho), (1_\rho \otimes w - \theta_\rho(w) \otimes 1_\rho) \rangle_{\text{End}(\iota_A)}^{(\rho, \rho)}$. Making use of the previous lemma shows that the product is zero, so the two objects must be equal as the $\text{End}(\iota_A)$ inner product is non-degenerate. Also notice that the right hand side of $\theta_\rho(w) \otimes 1_\rho = 1_\rho \otimes w$ does not depend on the choice of solutions R, \bar{R} , thus the isomorphism θ_ρ must be independent as well. \blacksquare

REMARK 2.17 In the same way we have an expression for $\theta_\rho^{-1} : \text{End}(\iota_B) \rightarrow \text{End}(\iota_A)$ with $\theta_\rho^{-1}(z) := \frac{\langle z \otimes 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)}^{(\rho, \rho)}}{\langle 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)}^{(\rho, \rho)}}$. Also note that $\theta_\rho = \theta_{\bar{\rho}}^{-1}$ and $\theta_{\rho'} \circ \theta_\rho = \theta_{\rho' \otimes \rho}$ (when the composition of arrows is defined).

REMARK 2.18 In the preceding proposition we have supposed $S_l(\rho) = \Omega_A$ and $S_r(\rho) = \Omega_B$. In the general case an analogous isomorphism holds for the subalgebras $E_{S_l(\rho)} \otimes \text{End}(\iota_A)$ and $E_{S_r(\rho)} \otimes \text{End}(\iota_B)$, which will be denoted by the same symbol $\theta_\rho : E_{S_l(\rho)} \otimes \text{End}(\iota_A) \rightarrow E_{S_r(\rho)} \otimes \text{End}(\iota_B)$. The same way, θ^{-1*} will indicate the homeomorphism between the two subspaces $S_l(\rho) \subset \Omega_A$ and $S_r(\rho) \subset \Omega_B$.

We now describe the behaviour of the \otimes product.

We begin with a remark about the supports of continuous sections.

LEMMA 2.19 Let $T \in \text{Hom}(\rho, \sigma)$, for ρ, σ two generic 1-arrows. Then support $T \subset S_l(\rho) \cap S_l(\sigma)$.

Proof. We have $T = 1_\sigma \circ T \circ 1_\rho = (1_\sigma \otimes E_{S_l(\sigma)}) \circ T \circ (1_\rho \otimes E_{S_l(\rho)}) = T \otimes E_{S_l(\sigma)} \otimes E_{S_l(\rho)}$, where in the second equality we have used Corollary 1.15. \blacksquare

REMARK 2.20 The same way we have $T = E_{S_r(\sigma)} \otimes E_{S_r(\rho)} \otimes T$.

Let $T, T' \in (C \xleftarrow{\rho'} B, C \xleftarrow{\sigma'} B)$ and $B \xleftarrow{\rho} A, D \xleftarrow{\rho''} C$. Then we have the following

COROLLARY 2.21 If $S_l(\sigma') \cap S_l(\rho') \cap S_r(\rho) = \emptyset$, then $T \otimes 1_\rho = 0$. Analogously, if $S_r(\sigma') \cap S_r(\rho') \cap S_l(\rho'') = \emptyset$, then $1_{\rho''} \otimes T = 0$.

For the rest of this section, we will consider only centrally balanced 1-arrows. This is not a real limitation, as we have supposed Assumption 2.14 to hold.

PROPOSITION 2.22 For $\theta_\rho^{-1*}(\omega) \in S_l(\rho')$, $(T \otimes 1_\rho)|_\omega = (T' \otimes 1_\rho)|_\omega$ iff $T|_{\theta_\rho^{-1*}(\omega)} = T'|_{\theta_\rho^{-1*}(\omega)}$.

The same way, for $\theta_{\rho'}^{-1*}(\alpha) \in S_l(\rho'')$, $(1_{\rho''} \otimes T)|_\alpha = (1_{\rho''} \otimes T')|_\alpha$ iff $T|_\alpha = T'|_\alpha$.

Proof. We fix solutions of the conjugation equations for ρ, ρ', ρ'' . For the product of the 1-arrows we take the product of the solutions. For example: $R_{\rho' \otimes \rho} := (1_{\bar{\rho}} \otimes R_{\rho'} \otimes 1_\rho) \circ R_\rho$, $\bar{R}_{\rho' \otimes \rho} := (1_{\rho'} \otimes \bar{R}_\rho \otimes 1_{\bar{\rho}'}) \circ \bar{R}_{\rho'}$.

$(T \otimes 1_\rho)|_\omega = (T' \otimes 1_\rho)|_\omega$ if and only if

$$\begin{aligned} 0 &= \langle (T - T') \otimes 1_\rho, (T - T') \otimes 1_\rho \rangle_{\text{End}(\iota_A)}^{(\rho' \otimes \rho, \sigma' \otimes \rho)}|_\omega \\ &= \theta_\rho^{-1}(\langle (T - T'), (T - T') \rangle_{\text{End}(\iota_B)}^{(\rho', \sigma')})|_\omega \times \langle 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)}^{(\rho, \rho)}|_\omega. \end{aligned}$$

Thus $T|_{\theta_\rho^{-1*}(\omega)} = T'|_{\theta_\rho^{-1*}(\omega)}$, as $\langle 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)}^{(\rho, \rho)}|_\omega \neq 0$.

The proof of the second statement is analogous. ■

We summarise the situation as follows. Associated to any non zero centrally balanced 1-arrow (say, $B \xleftarrow{\rho} A$) we have homeomorphic subspaces $S_l(\rho) \subset \Omega_A$ and $S_r(\rho) \subset \Omega_B$. The conjugation relations give explicit expressions of the isomorphism, depending on the choice of the 1-arrow ρ , but not on the choice of the solutions to the conjugation equations.

We are now in the position to give the following

DEFINITION 2.23 We define a \circ and \otimes product on the fibres: Let $S \in \text{Hom}(\rho, \sigma)$, $T \in \text{Hom}(\eta, \rho)$, $Q \in (\rho', \sigma')$; then

$$S_\omega \circ T_\omega := (S \circ T)_\omega;$$

$$Q_{\theta_\rho^{-1*}(\omega)} \otimes S_\omega := (Q \otimes S)_\omega, \text{ for } \omega \in S_l(\rho) \cap S_l(\sigma).$$

The consistency of these definitions is ensured by Propositions 2.3 and 2.22.

Notice that the \circ composition is defined between fibres with the same base point, while the \otimes composition is defined for fibres with base points in distinct topological spaces, the correspondence given by the homeomorphism between the supports. Thus, we can think of the \circ product as composing 2-arrows fibrewise and the \otimes product acting on the fibre structure by “gluing” the two supports of the bundles by means of the homeomorphism θ^{-1*} .

As a consequence we have

COROLLARY 2.24 Let $\omega \in S_l(\rho)$, then the map

$$T|_{\theta_\rho^{-1*}(\omega)} \mapsto T|_{\theta_\rho^{-1*}(\omega)} \otimes 1_{\rho|_\omega}$$

is injective;

the same way, let $\theta_{\rho'}^{-1*}(\alpha) \in S_l(\rho'')$, then the map

$$T|_\alpha \mapsto 1_{\rho''|_{\theta_{\rho'}^{-1*}(\alpha)}} \otimes T|_\alpha$$

is injective.

COROLLARY 2.25 *Let $\omega \in S_l(\rho)$. Then $R|_\omega$ and $\bar{R}|_{\theta_\rho^{-1*}(\omega)}$ satisfy the conjugation relations:*

$$\begin{aligned} (\bar{R}|_{\theta_\rho^{-1*}(\omega)} \otimes 1_{\rho|_\omega}) \circ (1_{\rho|_\omega} \otimes R|_\omega) &= 1_{\rho|_\omega}; \\ (R|_\omega \otimes 1_{\bar{\rho}|_{\theta_\rho^{-1*}(\omega)}}) \circ (1_{\bar{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes \bar{R}|_{\theta_\rho^{-1*}(\omega)}) &= 1_{\rho|_{\theta_\rho^{-1*}(\omega)}}. \end{aligned}$$

COROLLARY 2.26 *The map \bullet induces a conjugate linear isomorphism, which we will indicate with the same symbol, between the fibres $\bullet : \text{Hom}(\rho, \sigma)|_\omega \rightarrow \text{Hom}(\bar{\rho}, \bar{\sigma})|_{\theta_\rho^{-1*}(\omega)}$.*

COROLLARY 2.27 *Let $X|_\omega \in \text{End}(\rho)|_\omega$ be positive. Then the following inequality holds in $\text{End}(\rho)|_\omega$: $X|_\omega \leq (\bar{R}^* \circ \bar{R})|_{\theta_\rho^{-1*}(\omega)} \otimes 1_{\rho|_\omega} \otimes (R|_\omega^* \circ (1_{\bar{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes X|_\omega) \circ R|_\omega)$.*

These assertions are the “local” version of the ones already encountered in the introduction. For example Corollary 2.27 is valid as long as $X|_\omega$ is positive, independently of the value X in other points (thus, X need not be positive in $\text{End}(\rho)$).

LEMMA 2.28 *Let $T \in (B \xleftarrow{\rho} A, B \xleftarrow{\sigma} A)$, with ρ and σ centrally balanced 1-arrows. Then $\forall \omega \in \Omega_A \mid T_\omega \neq 0$ the homeomorphisms between Ω_A and Ω_B induced by ρ and σ coincide, i.e. $\theta_\rho^{*-1}(\omega) = \theta_\sigma^{*-1}(\omega)$.*

Proof. For any $w \in \text{End}(\iota_A)$ we have

$$T \otimes w = T \circ (1_\rho \otimes w) = T \circ (\theta_\rho(w) \otimes 1_\rho) = \theta_\rho(w) \otimes T$$

and in the same way we have

$$T \otimes w = (1_\sigma \otimes w) \circ T = \theta_\sigma(w) \otimes 1_\sigma \circ T = \theta_\sigma(w) \otimes T.$$

In particular

$$(\theta_\rho(w) \otimes T)|_\omega = ((\theta_\sigma(w) \otimes T)|_\omega)$$

which by Proposition 2.22 is equivalent to

$$\theta_\rho(w)|_{\theta_\rho^{*-1}(\omega)} \otimes T|_\omega = \theta_\sigma(w)|_{\theta_\sigma^{*-1}(\omega)} \otimes T|_\omega,$$

which in turn gives $\theta_\rho(w)|_{\theta_\rho^{*-1}(\omega)} = \theta_\sigma(w)|_{\theta_\sigma^{*-1}(\omega)}$. As this must hold for any $w \in \text{End}(\iota_A)$ we have $\theta_\rho^{*-1}(\omega) = \theta_\sigma^{*-1}(\omega)$. \blacksquare

We show now that starting from a 2- C^* -category \mathcal{C} , for each $\omega_0 \in \Omega_A$, where A is an arbitrary object, it is possible to construct a 2- C^* -category, which we will indicate by $\mathcal{C}^{\omega_0, A}$, with simple units. Consider the full sub 2- C^* -category of \mathcal{T} of C generated by centrally balanced 1-arrows, their products

and their sub1-arrows. This way to each $B \xleftarrow{\rho} A \in \mathcal{T}$ will be associated a homeomorphism $\theta_\rho^{*-1} : S_l(\rho) \rightarrow S_r(\rho)$.

Now define:

- as objects the set $\{\omega_B \in \Omega_B, \forall B \in \mathcal{C}\}$ such that there exists a $B \xleftarrow{\rho} A \in \mathcal{T}$ with $\omega_0 \in S_l(\rho), \omega_B \in S_r(\rho)$ and $\omega_B = \theta_\rho^{*-1}(\omega_0)$,
- as 1-arrows for any ω_B, ω_C as above a $\omega_C \xleftarrow{\sigma} \omega_B$ in correspondence to any $C \xleftarrow{\sigma} B$ with $\omega_B \in S_l(\sigma), \omega_C \in S_r(\sigma)$, verifying $\theta_\sigma^{*-1}(\omega_B) = \omega_C$,
- as 2-arrows, for any $\omega_C \xleftarrow{\sigma} \omega_B, \omega_C \xleftarrow{\eta} \omega_B$ as above, $\text{Hom}(\omega_C \xleftarrow{\sigma} \omega_B, \omega_C \xleftarrow{\eta} \omega_B)$ to be the set $\{T|_{\omega_B}, \forall T \in \text{Hom}(\sigma, \eta)\}$.

REMARK 2.29 By the preceding lemma we see that for $\omega_C \xleftarrow{\sigma} \omega_B, \omega'_C \xleftarrow{\eta} \omega_B, \omega_C \neq \omega'_C \forall T \in \text{Hom}(\sigma, \eta) T|_{\omega_B} = 0$, i.e. we don't loose any information by considering ω_C and ω'_C as distinct objects in the new category.

We define the \circ and \otimes products of Definition 2.23 as the operations of our new category and we endow the spaces of 2-arrows with the fibre C^* -norm introduced above, thus obtaining a $2\text{-}C^*$ -category. For each object ω_B we have the 1-unit $\omega_B \xleftarrow{\iota_{\omega_B}} \omega_B$ corresponding to ι_B . For each 1-arrow $\omega_C \xleftarrow{\sigma} \omega_B$ we have the 2-unit $1_{\sigma|_{\omega_B}}$. As

$$\text{End}(\omega_B \xleftarrow{\iota_{\omega_B}} \omega_B) \cong \text{End}(\iota_B)_{\omega_B} \cong \mathbb{C},$$

we see that each 1-unit is simple. Corollary 2.25 ensures that this category is closed under conjugation. Closure with respect to projections is not automatically ensured. For example, if $P^{\omega_B} \in (\omega_C \xleftarrow{\sigma} \omega_B, \omega_C \xleftarrow{\sigma} \omega_B)$ is a projection, there does not necessarily exist a projection (thus a corresponding 1-arrow) P in $\text{Hom}(\sigma, \sigma)$ such that $P|_{\omega_B} = P^{\omega_B}$. We will consider the completion under projections of the above category, and denote it by $\mathcal{C}^{\omega_0, A}$.

Note that in $\mathcal{C}^{\omega_0, A}$ several points $\omega_B, \omega'_B \dots$ of Ω_B may appear as distinct objects. In fact, as the various maps $\theta_\rho^{*-1} : S_l(\rho) \rightarrow S_r(\rho)$ are invertible, we see that each point ω_0 determines an orbit in the spaces $\Omega_A, \Omega_B, \dots$, and that the construction leading to $\mathcal{C}^{\omega_0, A}$ depends only on the choice of one of these (disjoint) orbits.

EXAMPLE 2.30 Let $B \xleftarrow{\rho} A, A \xleftarrow{\bar{\rho}} B$, be a pair of centrally balanced, conjugate 1-arrows with $E_{S_l(\rho)} = \Omega_A$ and $E_{S_r(\rho)} = \Omega_B$. We can consider the full sub $2\text{-}C^*$ -category generated by these two elements, i.e. their compositions $\rho, \rho \otimes \bar{\rho}, \rho \otimes \bar{\rho} \otimes \rho \dots$ and their sub-1-arrows. This is the categorical analogue of Jones' basic construction. We can take the sequence of algebras

$$\text{End}(\rho), \text{End}(\rho \otimes \bar{\rho}), \text{End}(\rho \otimes \bar{\rho} \otimes \rho), \dots$$

and realise a sequence of injective inclusions as follows:

$$\text{End}(\rho) \ni X \hookrightarrow X \otimes 1_{\bar{\rho}} \in \text{End}(\rho \otimes \bar{\rho}),$$

$$\text{End}(\rho \otimes \bar{\rho}) \ni Z \hookrightarrow Z \otimes 1_{\rho} \in \text{End}(\rho \otimes \bar{\rho} \otimes \rho), \dots$$

We choose $\omega_0 \in \Omega_A$ and construct $\mathcal{C}^{\omega_0, A}$ as above. In this case we have only two objects, namely ω_0 and $\theta_{\rho}^{*-1}(\omega_0)$, as $\theta_{\bar{\rho}}^* = \theta_{\rho}^{*-1}$ implies $\theta_{\rho \otimes \bar{\rho}}^{*-1} = \theta_{\rho}^{*-1} \circ \theta_{\bar{\rho}}^{*-1} = id$. Thus, for example,

$$\theta_{\rho \otimes \bar{\rho} \otimes \rho}^{*-1}(\omega_0) \xleftarrow{\rho \otimes \bar{\rho} \otimes \rho} \omega_0 = \theta_{\rho}^{*-1}(\omega_0) \xleftarrow{\rho \otimes \bar{\rho} \otimes \rho} \omega_0,$$

and analogously for the other 1-arrows of $\mathcal{A}^{\omega_0, A}$. As in the case of subfactors, we have a sequence of inclusions of finite dimensional C^* -algebras:

$$\text{End}(\rho)_{\omega_0} \ni X|_{\omega_0} \hookrightarrow X|_{\omega_0} \otimes 1_{\bar{\rho}|_{\theta_{\rho}^{*-1}(\omega_0)}} \in \text{End}(\rho \otimes \bar{\rho})_{\theta_{\rho}^{*-1}(\omega_0)}$$

$$\text{End}(\rho \otimes \bar{\rho})_{\theta_{\rho}^{*-1}(\omega_0)} \ni Z|_{\theta_{\rho}^{*-1}(\omega_0)} \hookrightarrow Z|_{\theta_{\rho}^{*-1}(\omega_0)} \otimes 1_{\rho|_{\omega_0}} \in \text{End}(\rho \otimes \bar{\rho} \otimes \rho)_{\omega_0} \dots$$

REMARK 2.31 A particular case is that of an $\text{End}(\iota)$ -linear tensor C^* -category \mathcal{T} , where ι is the unit element. $\text{End}(\iota)$ -linear means that for any $k \in \text{End}(\iota) \cong C(\Omega)$ and for any $\rho \in \mathcal{T}$, one has $\rho \otimes k = k \otimes \rho$. Clearly our Assumption 2.14 holds, namely every object of the tensor category is centrally balanced: for each object ρ the homeomorphism $\theta_{\rho}^{-1*} : S_{\iota}(\rho) \rightarrow S_r(\rho) = S_{\iota}(\rho)$ is the identity map. The above construction assumes a clear form in this case, which we are tempted to describe as that of a “bundle of tensor C^* -categories with simple units” over the topological space Ω :

PROPOSITION 2.32 *Let \mathcal{T} be an $\text{End}(\iota)$ -linear tensor C^* -category closed for conjugation, sub-objects and direct sums. Let Ω be the compact Hausdorff topological space associated to $\text{End}(\iota) \cong C(\Omega)$. Then for each $\omega \in \Omega$ there is an associated tensor C^* -category \mathcal{T}_{ω} with simple unit object, closed for conjugation and direct sums, the fibre category at the point ω . The arrows of the original category \mathcal{T} can be viewed as continuous sections taking values in the Hom spaces of the fibre categories. The $\circ, \otimes, *$ operations, as well as the conjugation operation of \mathcal{T} , preserve the fibre structure.*

3 Standard solutions

In this section we will introduce a particular class of solutions to the conjugation equations.

We begin by recalling the analogous definition and some basic facts concerning the case of simple units, contained in [16]. So, for the moment, we suppose that ρ is a 1-arrow going from A to B and that $\text{End}(\iota_A) \cong \mathbb{C}$, $\text{End}(\iota_B) \cong \mathbb{C}$.

Let $\bar{\rho}$ be a conjugate 1-arrow going from B to A . The main difference (and simplification) from the general case is that the algebra $\text{End}(\rho)$ and its

isomorphic $\text{End}(\bar{\rho})$ are finite dimensional C^* -algebras, i.e. a finite direct sum of matrix algebras. It is thus possible to decompose the unit arrow 1_ρ as a sum of minimal projections e_i in the algebra $\text{End}(\rho)$. To each of these minimal projections will correspond an irreducible ρ_i , i.e. $\text{End}(\rho_i) \cong \mathbb{C}$, and we think of ρ as a direct sum $\oplus_i \rho_i$, i.e. there exists a complete family of isometric 2-arrows $W_i \in \text{Hom}(\rho_i, \rho)$ s.t. $W_i \circ W_i^* = e_i$, $W_i^* \circ W_i = 1_{\rho_i}$. Two projections e_i, e_j dominated by the same minimal central projection in $\text{End}(\rho)$ will lead to equivalent irreducible 1-arrows, i.e. there will exist a unitary $V_{i,j} \in \text{Hom}(\rho_j)$.

For irreducible ρ , i.e. $\text{End}(\rho) \cong \mathbb{C}$, we give the following definition (notice that by duality, $\bar{\rho}$ is irreducible too):

DEFINITION 3.1 *Let $\rho, \bar{\rho}$ be irreducible. R, \bar{R} are said to be a standard solution if $R^* \circ R = \bar{R}^* \circ \bar{R}$.*

Let now ρ be not necessarily irreducible and $\oplus_i \rho_i$ its decomposition into irreducibles. Let $\{\bar{\rho}_i\}$ be irreducibles, each conjugate to its corresponding ρ_i and R_i, \bar{R}_i standard solutions for each of these couples.

Then $\oplus_i \bar{\rho}_i$ and ρ are conjugate and $\oplus_i R_i, \oplus_i \bar{R}_i$ is a solution.

DEFINITION 3.2 *Let $\rho = \oplus_i \rho_i$ and $\bar{\rho} = \oplus_i \bar{\rho}_i$ be conjugate. A solution of the form $\oplus_i R_i, \oplus_i \bar{R}_i$, where the R_i and \bar{R}_i are irreducible and standard as defined above, is called a standard solution.*

As we will see, standard solutions are uniquely defined up to a unitary in $\text{End}(\rho)$.

In this context standard solutions always exist:

PROPOSITION 3.3 *Let ρ and $\bar{\rho}$ be conjugate 1-arrows between objects A and B , with $\text{End}(\iota_A) \cong \mathbb{C}$ and $\text{End}(\iota_B) \cong \mathbb{C}$. Then standard solutions always exist.*

Proof. Choose an arbitrary solution R', \bar{R}' . Decompose ρ into a sum of irreducibles $\oplus_i \rho_i$ by means of a complete family of orthogonal projections $\{e_i\}$ in $\text{End}(\rho)$. The corresponding elements $e_i^\bullet \in \text{End}(\bar{\rho})$ will be a complete family of disjoint idempotents (i.e. $e_i^\bullet e_j^\bullet = \delta_{i,j} e_i^\bullet$), as the \bullet operation is an anti-isomorphism between the algebras $\text{End}(\rho)$ and $\text{End}(\bar{\rho})$. But they will fail, in general, to be self adjoint. Nevertheless there will exist an invertible $A \in \text{End}(\bar{\rho})$ such that $\{\bar{e}_i := A e_i^\bullet A^{-1}\}$ is a complete family of orthogonal projections. As before, we can decompose $\bar{\rho} = \oplus_i \bar{\rho}_i$, where each $\bar{\rho}_i$ corresponds to the projection \bar{e}_i . Taking now $R'' := (A \otimes 1_\rho) \circ R'$ and $\bar{R}'' := (1_\rho \otimes A^{-1*}) \circ \bar{R}'$ as solutions, we see that each pair ρ_i and $\bar{\rho}_i$ is a couple of conjugates, with $R_i'' := (\bar{e}_i \otimes e_i) \circ R''$, $\bar{R}_i'' := (e_i \otimes \bar{e}_i) \circ \bar{R}''$ as solutions.

Furthermore each pair R_i'', \bar{R}_i'' is determined up to a constant as ρ_i and $\bar{\rho}_i$ are irreducible. We rescale each couple by a constant so to have the equality $R_i''^* \circ R_i'' = \bar{R}_i''^* \circ \bar{R}_i''$. This amounts to multiplying the invertible element $A \in \text{End}(\bar{\rho})$ by a diagonal element D in the same algebra. We set $R := (D \otimes 1_\rho) \circ R''$

and $\bar{R} := (1_\rho \otimes D^{-1}) \circ \bar{R}''$ and $R_i := (\bar{e}_i \otimes e_i) \circ R$, $\bar{R}_i := (e_i \otimes \bar{e}_i) \circ \bar{R}$. It is easily verified that $R = \oplus_i R_i$ and $\bar{R} = \oplus_i \bar{R}_i$. \blacksquare

The number $R_i^* \circ R_i = \bar{R}_i^* \circ \bar{R}_i$ is called the dimension of the irreducible ρ_i and depends only on the equivalence class of the ρ_i . In fact, suppose ρ_i and ρ_j are equivalent, i.e. there exists a partial isometry $V_{i,j} \in \text{End}(\rho)$ such that $V_{i,j}^* \circ V_{i,j} = e_i$ and $V_{i,j} \circ V_{i,j}^* = e_j$ (in other words, e_i and e_j have same central support). Then, by duality, the same will hold for $\bar{\rho}_i$ and $\bar{\rho}_j$, with a partial isometry $\bar{V}_{i,j} \in \text{End}(\bar{\rho})$. Take $(\bar{V}_{i,j} \otimes V_{i,j}) \circ R_i$ and $(V_{i,j} \otimes \bar{V}_{i,j}) \circ \bar{R}_i$ as a solution for ρ_j and $\bar{\rho}_j$. This solution can differ from R_j, \bar{R}_j only by a invertible element in $\text{End}(\rho_j) = \mathbb{C}$, i.e. $(\bar{V}_{i,j} \otimes V_{i,j}) \circ R_i = \lambda R_j$ and $(V_{i,j} \otimes \bar{V}_{i,j}) \circ \bar{R}_i = \lambda^{-1} \bar{R}_j$. But this implies $(R_i^* \circ R_i)(\bar{R}_i^* \circ \bar{R}_i) = (R_j^* \circ R_j)(\bar{R}_j^* \circ \bar{R}_j)$ and, because of the normalization we have chosen above, $R_i^* \circ R_i = R_j^* \circ R_j$.

The number $R^* \circ R = \sum_i R_i^* \circ R_i$ is the dimension of the object ρ . It is additive with respect to direct sums.

The class of standard solution defines a trace on the algebra $\text{End}(\rho)$ (and in an analogous way on the algebra $\text{End}(\bar{\rho})$). Consider an element $S \in \text{End}(\rho)$. Thinking of $\text{End}(\rho)$ as a direct sum of matrix algebras, we indicate by $S_{i,j}$ the matrix elements of S corresponding to a representation given by the projectors $\{e_i\}$. Thus $(\bar{R}^* \circ (S \otimes 1_{\bar{\rho}}) \circ \bar{R}) = \sum_i S_{i,i} \bar{R}_i^* \circ \bar{R}_i = \sum_i S_{i,i} R_i^* \circ R_i = R^* \circ (1_{\bar{\rho}} \otimes S) \circ R$ which is a trace, as the dimensions $R_i^* \circ R_i$ depend only on the central supports of the e_i .

Notice that we have shown that for a standard solution the following holds:

$$R^* \circ (1_{\bar{\rho}} \otimes S) \circ R = \bar{R}^* \circ (S \otimes 1_{\bar{\rho}}) \circ \bar{R}, \quad \forall S \in \text{End}(\rho).$$

This can be used as an equivalent definition for standardness (see [16]).

REMARK 3.4 We can summarize the situation as follows: fix a faithful normalised trace tr (i.e. $tr(1) = 1$) on the algebra $\text{End}(\rho)$. A generic solution R', \bar{R}' to the conjugation relations will induce faithful functionals $\Phi'(S) := R' \circ (1_\rho \otimes S) \circ R' = tr(DKS)$, $\Psi'(S) := \bar{R}' \circ (S \otimes 1_{\bar{\rho}}) \circ \bar{R}' = tr(DK^{-1}S)$, where D is a positive invertible central element in $\text{End}(\rho)$, whose trace is the dimension of ρ , and K is a positive invertible element of $\text{End}(\rho)$. Taking $R := (1_{\bar{\rho}} \otimes K^{-1}) \circ R'$ and $\bar{R} := (K \otimes 1_{\bar{\rho}}) \circ \bar{R}'$ will give a standard solution. Notice that the element K is uniquely defined by the original choice R', \bar{R}' . The arbitrariness of the choice of the trace is expressed by the central element D .

We now drop the hypothesis of simple units. We begin by giving the definition of standardness for centrally balanced 1-arrows.

DEFINITION 3.5 Let $\rho, \bar{\rho}$ be centrally balanced. Let R, \bar{R} be a solution to the conjugation equations. We say R, \bar{R} to be standard if $\forall X \in \text{End}(\rho)$ the following holds:

$$1_\rho \otimes (R^* \circ (1_{\bar{\rho}} \otimes X) \circ R) = (\bar{R}^* \circ (X \otimes 1_{\bar{\rho}}) \circ \bar{R}) \otimes 1_\rho.$$

The following lemma shows that another appropriate name could have been “minimal”:

LEMMA 3.6 *Let R, \bar{R} be standard solutions for centrally balanced $\rho, \bar{\rho}$. Then for any other solution R', \bar{R}' we have: $(\bar{R}'^* \circ \bar{R}') \otimes 1_\rho \otimes (R'^* \circ R') \geq (\bar{R}^* \circ \bar{R}) \otimes 1_\rho \otimes (R^* \circ R)$.*

Proof. We have $R' = (1_{\bar{\rho}} \otimes X) \circ R$ and $\bar{R}' = (X^{-1*} \otimes 1_{\bar{\rho}}) \circ \bar{R}$ for some invertible $X \in \text{End}(\rho)$. Thus

$$\begin{aligned} (\bar{R}'^* \circ \bar{R}') \otimes 1_\rho \otimes (R'^* \circ R') &= \\ (\bar{R}^* \circ (X^{-1} \circ X^{-1*}) \otimes 1_{\bar{\rho}} \circ \bar{R}) \otimes 1_\rho \otimes (R^* \circ 1_{\bar{\rho}} \otimes (X^* \circ X) \circ R) &= \\ 1_\rho \otimes ((R^* \circ 1_{\bar{\rho}} \otimes (X^* \circ X) \circ R) \circ (R^* \circ 1_{\bar{\rho}} \otimes (X^{-1} \circ X^{-1*}) \circ R)) &= \end{aligned}$$

The claim is implied by the inequality

$$(R^* \circ 1_{\bar{\rho}} \otimes (X^* \circ X) \circ R) \circ (R^* \circ 1_{\bar{\rho}} \otimes (X \circ X^{-1*}) \circ R) \geq (R^* \circ R)^2,$$

which can be written equivalently

$$\langle X, X \rangle_{\text{End}(\iota_A)} \langle X^{-1*}, X^{-1*} \rangle_{(\iota_A, \iota_A)} \geq \langle 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)}^2$$

(where we have used the $\langle \cdot, \cdot \rangle_{\text{End}(\iota_A)}$ inner product defined by R and \bar{R}).

It is sufficient to prove this inequality for each $\omega \in \Omega_A$, which is easily done by means of a Cauchy-Schwarz argument: first rewrite $\langle X, X \rangle_{\text{End}(\iota_A)}$ and $\langle X^{-1*}, X^{-1*} \rangle_{\text{End}(\iota_A)}$ as $\langle (X^* \circ X)^{\frac{1}{2}}, (X^* \circ X)^{\frac{1}{2}} \rangle_{(\iota_A, \iota_A)}$ and $\langle (X^* \circ X)^{-\frac{1}{2}}, (X^* \circ X)^{-\frac{1}{2}} \rangle_{\text{End}(\iota_A)}$ respectively. Then

$$\begin{aligned} \langle (X \circ X^*)^{\frac{1}{2}}, (X \circ X^*)^{\frac{1}{2}} \rangle_{\text{End}(\iota_A)} \langle (X \circ X^*)^{-\frac{1}{2}}, (X \circ X^*)^{-\frac{1}{2}} \rangle_{\text{End}(\iota_A)} \\ \geq (\langle 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)})^2. \quad \blacksquare \end{aligned}$$

LEMMA 3.7 *Suppose that R, \bar{R} and R', \bar{R}' are two pairs of standard solutions. Then there exists a unitary $U \in \text{End}(\rho)$ such that $R' = (1_{\bar{\rho}} \otimes U) \circ R$ and $\bar{R}' = (U \otimes 1_{\bar{\rho}}) \circ \bar{R}$.*

Proof. As we have seen, there exists an invertible U satisfying $R' = (1_{\bar{\rho}} \otimes U) \circ R$ and $\bar{R}' = (U^{-1*} \otimes 1_{\bar{\rho}}) \circ \bar{R}$. We must prove that U is unitary.

By the definition of standardness we have

$$(\bar{R} \circ A \otimes 1_{\bar{\rho}} \circ \bar{R}) \otimes 1_\rho = 1_\rho \otimes (R^* \circ 1_{\bar{\rho}} \otimes A \circ R), \quad \forall A \in \text{End}(\rho).$$

This implies $(\bar{R} \circ (U^{-1} A U^{-1}) \otimes 1_{\bar{\rho}} \circ \bar{R}) \otimes 1_\rho = 1_\rho \otimes (R^* \circ 1_{\bar{\rho}} \otimes (U^* A U^{-1*}) \circ R)$. As R' and \bar{R}' are standard solutions as well, we have

$$(\bar{R}' \circ (U^{-1} A U^{-1*}) \otimes 1_{\bar{\rho}} \circ \bar{R}') \otimes 1_\rho = 1_\rho \otimes (R'^* \circ 1_{\bar{\rho}} \otimes (U^* A U) \circ R), \quad \forall A \in \text{End}(\rho).$$

This implies

$$\langle U, AU \rangle_{\text{End}(\iota_A)} = \langle U^{-1*}, AU^{-1*} \rangle_{\text{End}(\iota_A)} \quad \forall A \in \text{End}(\rho).$$

In particular $\langle U, U \rangle_{\text{End}(\iota_A)} = \langle U^{-1*}, U^{-1*} \rangle_{\text{End}(\iota_A)}$. R' and \bar{R}' also satisfy the minimality condition of the preceding lemma, i.e. $(\bar{R}'^* \circ R'^*) \otimes 1_\rho \otimes (R'^* \circ R') = (\bar{R}^* \circ \bar{R}) \otimes 1_\rho \otimes (R^* \circ R)$. In other words we have $\langle U, U \rangle_{\text{End}(\iota_A)} = \langle U^{-1*}, U^{-1*} \rangle_{\text{End}(\iota_A)} = \langle 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)}$ (where we are using as before the $\langle \cdot, \cdot \rangle_{\text{End}(\iota_A)}$ inner product relative to the R, \bar{R} solution). Noticing that

$$\begin{aligned} \langle U^*U, U^*U \rangle_{\text{End}(\iota_A)} &= \langle UU^*U, U \rangle_{\text{End}(\iota_A)} \\ &= \langle U^{-1*}U^*U, U^{-1*} \rangle_{\text{End}(\iota_A)} = \langle 1_\rho, 1_\rho \rangle_{\text{End}(\iota_A)} \end{aligned}$$

we see that the product $\langle (1 - U^*U), (1 - U^*U) \rangle_{\text{End}(\iota_A)}$ is zero.

Thus $U = U^{-1*}$, as the $\text{End}(\iota_A)$ -valued inner product is non-degenerate. ■

REMARK 3.8 In an analogous way one can show that there exists a unitary $\bar{U} \in (\bar{\rho}, \bar{\rho})$ such that $R' = \bar{U} \otimes 1_\rho \circ R$ and $\bar{R}' = 1_\rho \otimes \bar{U} \circ \bar{R}$.

PROPOSITION 3.9 *Let R, \bar{R} be a standard solution for $\rho, \bar{\rho}$. Then the associated inner product $\langle \cdot, \cdot \rangle_{\text{End}(\iota_A)}$ is tracial, i.e.*

$$\langle ST, 1_\rho \rangle_{\text{End}(\iota_A)} = \langle TS, 1_\rho \rangle_{\text{End}(\iota_A)} \quad \forall S, T \in \text{End}(\rho).$$

Proof. It suffices to prove

$$\langle U^*SU, 1_\rho \rangle_{\text{End}(\iota_A)} = \langle S, 1_\rho \rangle_{\text{End}(\iota_A)}$$

for any unitary $U \in \text{End}(\rho)$, $\forall S \in \text{End}(\rho)$. We have

$$\begin{aligned} \langle U^*SU, 1_\rho \rangle_{\text{End}(\iota_A)} &= R^*(1_{\bar{\rho}} \otimes U^*SU) \circ R \\ &= R^* \circ (1_{\bar{\rho}} \otimes U^*) \circ (1_{\bar{\rho}} \otimes S) \circ (1_{\bar{\rho}} \otimes U) \circ R. \end{aligned}$$

But $R' := (1_{\bar{\rho}} \otimes U) \circ R$, $\bar{R}' := (U \otimes 1_{\bar{\rho}}) \circ \bar{R}$ is still a standard solution, so there exists a unitary $\bar{U} \in (\bar{\rho}, \bar{\rho})$ such that $R' = (\bar{U} \otimes 1_\rho) \circ R$ and $\bar{R}' = (1_\rho \otimes \bar{U}) \circ \bar{R}$. Thus

$$\begin{aligned} \langle U^*SU, 1_\rho \rangle_{\text{End}(\iota_A)} &= R^* \circ (\bar{U}^* \otimes 1_\rho) \circ (1_{\bar{\rho}} \otimes S) \circ (\bar{U} \otimes 1_\rho) \circ R \\ &= R^* \circ (1_{\bar{\rho}} \otimes S) \circ R = \langle S, 1_\rho \rangle_{\text{End}(\iota_A)}. \end{aligned}$$

■

REMARK 3.10 Analogously one proves that the $\text{End}(\iota_B)$ inner product is tracial too.

LEMMA 3.11 *Let ρ and σ be two centrally balanced 1-arrows with standard solutions R_ρ, \bar{R}_ρ and R_σ, \bar{R}_σ respectively. Then the product solution, defined by*

$$R_{\sigma \otimes \rho} := 1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho \circ R_\rho, \quad \bar{R}_{\sigma \otimes \rho} := 1_\sigma \otimes \bar{R}_\rho \otimes 1_{\bar{\sigma}} \circ \bar{R}_\sigma,$$

is standard.

Proof. For every $A \in \text{End}(\sigma \otimes \rho)$ we have:

$$\begin{aligned} & 1_{\sigma \otimes \rho} \otimes (R_{\sigma \otimes \rho}^* \circ (1_{\bar{\rho} \otimes \bar{\sigma}} \otimes A) \circ R_{\sigma \otimes \rho}) = \\ & 1_\sigma \otimes 1_\rho \otimes (R_\rho^* \circ (1_{\bar{\rho}} \otimes R_\sigma^* \otimes 1_\rho) \circ (1_{\bar{\rho}} \otimes 1_{\bar{\sigma}} \otimes A) \circ (1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho) \circ R_\rho). \end{aligned}$$

Using the standard solution property of R_ρ, \bar{R}_ρ on the element $R_\sigma^* \otimes 1_\rho \circ (1_{\bar{\sigma}} \otimes A) \circ R_\sigma \otimes 1_\rho \in \text{End}(\rho)$ we get:

$$\begin{aligned} & 1_\sigma \otimes 1_\rho \otimes (R_\rho^* \circ (1_{\bar{\rho}} \otimes R_\sigma^* \otimes 1_\rho) \circ (1_{\bar{\rho}} \otimes 1_{\bar{\sigma}} \otimes A) \circ (1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho) \circ R_\rho) = \\ & 1_\sigma \otimes (\bar{R}_\rho^* \circ ((R_\sigma^* \otimes 1_\rho \otimes 1_{\bar{\sigma}} \otimes A \circ R_\sigma \otimes 1_\rho) \otimes 1_{\bar{\rho}}) \circ \bar{R}_\rho) \otimes 1_\rho = \\ & 1_\sigma \otimes (R_\sigma^* \otimes \bar{R}_\rho^* \circ (1_{\bar{\sigma}} \otimes A \otimes 1_\rho) \circ R_\sigma \otimes \bar{R}_\rho) \otimes 1_\rho. \end{aligned}$$

The same reasoning applied to the solution R_σ, \bar{R}_σ and the element $1_\sigma \otimes \bar{R}_\rho^* \circ (A \otimes 1_{\bar{\rho}}) \circ 1_\sigma \otimes \bar{R}_\rho \in \text{End}(\sigma)$ gives:

$$\begin{aligned} & 1_\sigma \otimes (R_\sigma^* \otimes \bar{R}_\rho^* \circ (1_{\bar{\sigma}} \otimes A \otimes 1_\rho) \circ R_\sigma \otimes \bar{R}_\rho) \otimes 1_\rho = \\ & (\bar{R}_\sigma^* \circ (1_\sigma \otimes \bar{R}_\rho^* \otimes 1_{\bar{\sigma}}) \circ (A \otimes 1_{\bar{\rho}} \otimes 1_{\bar{\sigma}}) \circ (1_\sigma \otimes \bar{R}_\rho \otimes 1_{\bar{\sigma}}) \circ \bar{R}_\sigma) \otimes 1_\sigma \otimes 1_\rho. \end{aligned}$$

Thus

$$1_{\sigma \otimes \rho} \otimes (R_{\sigma \otimes \rho}^* \circ (1_{\bar{\rho} \otimes \bar{\sigma}} \otimes A) \circ R_{\sigma \otimes \rho}) = (\bar{R}_{\sigma \otimes \rho}^* \circ (A \otimes 1_{\sigma \otimes \rho}) \circ \bar{R}_{\sigma \otimes \rho}) \otimes 1_{\sigma \otimes \rho}.$$

■

We now give the natural definition of standardness for a generic 1-arrow.

DEFINITION 3.12 *Let ρ and $\bar{\rho}$ be a direct sum of centrally balanced 1-arrows $\oplus_i \rho_i$ and $\oplus_i \bar{\rho}_i$ respectively, the decomposition given by complete sets of orthogonal partial isometries $\{W_i \in \text{Hom}(\rho_i, \rho)\}$ and $\{\bar{W}_i \in \text{Hom}(\bar{\rho}_i, \bar{\rho})\}$. A solution R, \bar{R} to the conjugation equations for $\rho, \bar{\rho}$ is said to be standard if it is of the form $\sum_i (\bar{W}_i \otimes W_i) \circ R_i$, $\sum_i (W_i \otimes \bar{W}_i) \circ \bar{R}_i$, where each couple R_i, \bar{R}_i is standard for $\rho_i, \bar{\rho}_i$, respectively.*

REMARK 3.13 With this definition, Proposition 3.9 and Lemmas 3.7, 3.11 are easily seen to hold in the general case too.

Thus we have the following

PROPOSITION 3.14 *The class of standard solutions is stable under the operations of direct sum, tensor product, projections and conjugation.*

The natural question is whether a choice of standard solution is available for all 1-arrows. In the rest of this chapter we will give some partial answers.

In order to do so, we first recollect some results concerning Banach and C^* -algebra bundles. As already mentioned, the notion of Banach bundle is more general than the more familiar notion of locally trivial bundle, even in the case of bundles with finite dimensional fibres. The following example shows the necessity for considering such a notion.

EXAMPLE 3.15 Consider the following tensor C^* -category (i.e. 2- C^* -category with one object): take $SU_n \times [0, 1]$, the trivial group bundle. $\Gamma(SU_n \times [0, 1])$ the group of continuous sections and $\mathcal{U} := \{\xi \in \Gamma(SU_n \times [0, 1]) \text{ s.t. } \xi_\omega = 1_n \text{ (the identity of } SU_n), \forall \omega \in [\frac{1}{2}, 1]\}$, a closed subgroup. Consider the trivial bundle $H \times [0, 1]$, where $H = \mathbb{C}^n$, together with the natural action of \mathcal{U} on it and denote it by ρ . Denote by $\rho^{\otimes n}$ the n th tensor product (fibre tensor product over the base space $[0, 1]$) of the same bundle with itself, with the natural action of \mathcal{U} . For $n = 0$ let $\rho^0 = \iota =: \mathbb{C} \times [0, 1]$, the trivial line bundle with the trivial action of \mathcal{U} .

The powers of ρ induce a tensor C^* -category, where $\text{Hom}(\rho^n, \rho^m)$ are sections of intertwining operators, i.e. continuous sections $S \in (H^n \times H^m) \times [0, 1]$ such that $S\rho^n(g)\xi = \rho^m(g)S\xi, \forall \xi \in \Gamma(H^n \times [0, 1]), \forall g \in \mathcal{U}$. It is easy to see that ι is the unit object in this tensor category. The conjugate object $\bar{\rho}$ is the conjugate fibre $\bar{H} \times [0, 1]$ with the conjugate action of \mathcal{U} . A standard solution for the conjugation equations is given by $R \in \text{Hom}(\iota, \bar{\rho}\rho) := \sum_i \bar{e}_i \otimes e_i$ and $\bar{R} \in \text{Hom}(\iota, \rho\bar{\rho}) := \sum_i e_i \otimes \bar{e}_i$, where e_i and \bar{e}_i are the constant sections given by the canonical basis for H and \bar{H} respectively.

Then $\text{End}(\rho) = \{S \in \Gamma(M_n \times [0, 1]) \text{ such that } S_\omega \in \mathbb{C}1_n \forall \omega \in [0, \frac{1}{2}]\}$. Thus the fibres of the Banach bundle $\text{End}(\rho)$ are of two types, \mathbb{C} for $\omega \in [0, \frac{1}{2}]$ and M_n for $\omega \in (\frac{1}{2}, 1]$.

We already know from the definition that $\forall \omega \in \Omega, \forall K^\omega \in \text{End}(\rho)_\omega, \exists A \in \text{End}(\rho)$ such that $A|_\omega = K^\omega$. In other words, each element of a single fibre can be extended to a continuous section defined on the whole base space. More can be said. The following lemma (whose proof can be found for example in [7]) will be useful in the sequel.

LEMMA 3.16 *Let F be a finite dimensional C^* -algebra, Ξ a C^* -algebra bundle over a normal topological space Ω and $\Omega \times F$ be the product (trivial) C^* -algebra bundle with fibre F . Let $\Phi : A \times F \rightarrow \Xi|_A$ be a C^* -algebra bundle embedding of the reduced bundles over a closed subset $A \subset \Omega$. Then there exists an open subset $U \supset A$ and an embedding $\Psi : U \times F \rightarrow \Xi|_U$ which extends Φ .*

This enables us to prove the following

LEMMA 3.17 *Let K^ω be a positive invertible element in the finite dimensional algebra $\text{End}(\rho)_\omega$. Then there exists an invertible $A \in \text{End}(\rho)$ and an open neighbourhood U_ω of ω such that $A|_\omega = K^\omega$ and $A|_{\omega'} = 1_{\rho|_{\omega'}}$, $\forall \omega' \notin U_\omega$.*

Proof. Take an open set $U_\omega \ni \omega$ and a C^* algebra bundle embedding $\Psi : U_\omega \times \text{End}(\rho)_\omega \rightarrow \text{End}(\rho)|_{U_\omega}$ extending the identity bundle embedding. Take a positive invertible section H in the product bundle $U_\omega \times \text{End}(\rho)_\omega$ which extends K^ω (for example the constant section). Now take a second open set $W \ni \omega$ such that $U_\omega \supset \overline{W}$ (we can do so, as the base space is normal) and a continuous complex-valued function f defined on U_ω such $f = 0$ out of W and $f(\omega) = 1$. Then $\Psi(\exp(f \ln H))$ will be a continuous section with value K^ω in ω and value $1_{\rho|_{\omega'}}$ for ω' out of W . Extending it with the identity section on the rest of the base space Ω we get a globally defined continuous section with the desired property. \blacksquare

We can now state a first result concerning standardness:

PROPOSITION 3.18 *Let $\rho, \bar{\rho}$ be centrally balanced. For each $\omega \in \Omega_A$ there exists a solution to the conjugation equations $R^\omega, \bar{R}^\omega \in \text{End}(\rho)$ such that $(1_\rho \otimes (R^{\omega*} \circ (1_{\bar{\rho}} \otimes X) \circ R^\omega))|_\omega = ((\bar{R}^{\omega*} \circ (X \otimes 1_\rho) \circ \bar{R}^\omega) \otimes 1_\rho)|_\omega$.*

Proof. Take a general solution R, \bar{R} , and suppose $\omega \in S_l(\rho)$, otherwise the case being trivial. As Corollary 2.25 points out, we can, mutatis mutandis, repeat the argument of the simple unit case for the algebras $\text{End}(\rho)_\omega$ and $\text{End}(\bar{\rho})_{\theta_\rho^{-1*}(\omega)}$ and the local solutions $R'_\omega, \bar{R}'_{\theta_\rho^{-1*}(\omega)}$. By the remarks after Proposition 3.3 we see that there exists a positive invertible $K^\omega \in \text{End}(\rho)_\omega$ such that

$$\begin{aligned} R^*_{|\omega} \circ (1_{\bar{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes (K^{\omega-1} \circ S_{|\omega} \circ K^{\omega-1}) \circ R_{|\omega} = \\ \bar{R}^*_{|\theta_\rho^{-1*}(\omega)} \circ ((K^\omega \circ S_{|\omega} \circ K^\omega \otimes 1_{\bar{\rho}|_{\theta_\rho^{-1*}(\omega)}}) \circ \bar{R}_{|\theta_\rho^{-1*}(\omega)} \end{aligned}$$

where we have used the \circ and \otimes products of fibre elements introduced in the preceding section. By the above lemma we can choose a positive invertible element $K \in \text{End}(\rho)$ such that $K|_\omega = K^\omega$. Let $R^\omega := (1_{\bar{\rho}} \otimes K^{-1}) \circ R$ and $\bar{R}^\omega := (K \otimes 1_{\bar{\rho}}) \circ \bar{R}$. \blacksquare

REMARK 3.19 The above proposition may be viewed as a local version of standardness. It implies that for each $\omega \in \Omega$, $R^{\omega*} \circ (1_{\bar{\rho}|_\omega} \otimes X|_\omega) \circ R|_\omega$ is a uniquely defined (up to normalization) trace on $\text{End}(\rho)_\omega$. It would be tempting to use this trace as a definition for a standard $\text{End}(\iota_A)$ -valued trace. Unfortunately in the general case the section K^ω is not a priori continuous, thus the above formula does not give a continuous trace, but only an upper semicontinuous one, as it is the inferior limit of a family of continuous functionals.

Nevertheless, we have the following

PROPOSITION 3.20 *Suppose $\text{End}(\rho)$ is a locally trivial bundle. Then a standard solution exists.*

Proof. Suppose, for simplicity, $\rho, \bar{\rho}$ to be centrally balanced (if not, decompose and consider each component separately, the associated $\text{Hom}(\rho_i, \rho_i)$ will still be locally trivial). If $\text{End}(\rho)$ is locally trivial it has constant fibre, i.e. for each point of the base space ω , $\text{End}(\rho)_\omega$ is isomorphic to a finite dimensional algebra F . We can choose a finite atlas of local charts, i.e. maps $\Theta_\alpha := U_\alpha \times F \rightarrow \text{End}(\rho)|_{U_\alpha}$ which are local isomorphisms of the trivial bundle $U_\alpha \times F$ onto the restriction of $\text{End}(\rho)$ over the open space $U_\alpha \subset \Omega$. The sets U_α form an open covering of Ω . Where two maps overlap we have transition functions, i.e. when for example $U_\alpha \cap U_\beta \neq \emptyset$, we have unitary sections in $W_{\alpha,\beta} \in (U_\alpha \cap U_\beta) \times F$ such that $\Theta_\alpha|_{U_\alpha \cap U_\beta} = W_{\alpha,\beta} \circ \Theta_\beta|_{U_\alpha \cap U_\beta} \circ W_{\alpha,\beta}^*$.

Fix a faithful trace tr on the algebra F . This defines a trace on each local chart Θ_α . As the transition functions are unitary sections, these local traces paste together into a continuous trace defined on the whole bundle. Take a generic solution R', \bar{R}' of the conjugation equations. As shown in the remark following Proposition 3.3 for each point ω of the base space we have $(R' \circ (1_{\bar{\rho}} \otimes S) \circ R')|_\omega = \text{tr}(D^\omega K^\omega S|_\omega)$ and $(\bar{R}' \circ (S \otimes 1_{\bar{\rho}}) \circ \bar{R}')|_{\bar{\rho}^{-1}*(\omega)} = \text{tr}(D^\omega K^{\omega^{-1}} S|_\omega)$ for a positive invertible K^ω and a positive invertible central D^ω in the fibre $\text{End}(\rho)_\omega$. Essentially we only have to prove that the section realized by these K^ω is a continuous section. Then there will be a corresponding element in $\text{End}(\rho)$ fulfilling our requirements.

But as the left hand sides of the above equations are continuous, and as the trace is continuous, this implies that both the sections $D^\omega K^\omega$ and $D^\omega K^{\omega^{-1}}$ are continuous sections in the locally trivial bundle $\text{End}(\rho)$. Thus K^ω is a continuous section, and a corresponding element $K \in \text{End}(\rho)$ exists. $R := (1_{\bar{\rho}} \otimes K^{-1}) \circ R'$, $\bar{R} := (K \otimes 1_{\bar{\rho}}) \circ \bar{R}'$ will be a standard solution. \blacksquare

4 Bundles of Hopf algebras

Finite irreducible subfactors of depth two are characterised by the action of finite dimensional Hopf algebras (see, for example, [15], [22]). This situation corresponds, in the context of 2- C^* -categories with simple units, to an irreducible 1-arrow ρ (i.e. $\text{End}(\rho) = \mathbb{C}1_\rho$) generating a 2- C^* -category of depth two (see below). The context can be generalised to the case of units with discrete and finite spectra (see [21]), leading to the appearance of Weak-Hopf algebras.

In the following we pursue the “orthogonal” direction, i.e. that of units with connected spectra, obtaining a continuous bundle of finite dimensional Hopf algebras in duality. We will follow the exposition given in [18], as it is closer to our context.

We begin by considering a centrally balanced 1-arrow $B \xleftarrow{\rho} A$ such that $\text{End}(\rho) = 1_\rho \otimes \text{End}(\iota_A) = \text{End}(\iota_B) \otimes 1_\rho$, with Ω_A and Ω_B connected homeo-

morphic spaces together with its conjugate $A \xleftarrow{\bar{\rho}} B$. We will call such ρ and $\bar{\rho}$ “irreducible”. We denote by \mathcal{C} the 2- C^* -category generated by compositions of ρ and $\bar{\rho}$ (i.e. $\rho, \rho \otimes \bar{\rho}, \rho \otimes \bar{\rho} \otimes \rho \dots$) and their projections.

In the case of categories with simple units, there is a well established notion of finite depth: the 2- C^* -category generated by ρ and $\bar{\rho}$ has finite depth n if the number of isomorphism classes of 1-arrows is finite (i.e. it’s rational) and all of them appear as sub-1-arrows of the first n products $\rho, \rho \otimes \bar{\rho} \dots$. For the sequence of inclusions $\text{End}(\rho) \otimes 1_{\bar{\rho}} \subset \text{End}(\rho \otimes \bar{\rho}) \dots$ this means that the corresponding principal part of the Bratteli diagram is finite, with depth n .

In the general case of non-simple units we say that ρ has finite depth n if for each $\omega \in \Omega_A$ the associated $\mathcal{C}^{\omega, A}$ has finite depth and the maximum depth among them is n . In particular we will be interested in the case of an irreducible ρ of depth two.

As ρ is irreducible, it is possible to find standard solutions R_ρ, \bar{R}_ρ . The same will be true for the products $\rho \otimes \bar{\rho}, \rho \otimes \bar{\rho} \otimes \rho \dots$, as the product solutions of standard solutions are still standard. In order to simplify the notation, we set

$$\mathcal{B} := (\rho \otimes \bar{\rho}, \rho \otimes \bar{\rho}), \quad \mathcal{A} := (\bar{\rho} \otimes \rho, \bar{\rho} \otimes \rho), \quad \mathcal{D} := (\rho \otimes \bar{\rho} \otimes \rho, \rho \otimes \bar{\rho} \otimes \rho).$$

\mathcal{A} and \mathcal{B} are in a natural way $C(\Omega_A)$ (respectively, $C(\Omega_B)$) bimodules, where left and right actions coincide (as $\theta_{\rho \otimes \bar{\rho}}$ and $\theta_{\bar{\rho} \otimes \rho}$ are the identity isomorphisms). As we have standard solutions $R_{\bar{\rho} \otimes \rho}, \bar{R}_{\bar{\rho} \otimes \rho}$ for $\bar{\rho} \otimes \rho$, the right and left $\text{End}(\iota_A)$ -valued inner products coincide as well and give a faithful (non normalised) trace Tr_A on \mathcal{A} :

$$a \in \mathcal{A}, \quad Tr_A(a) := \langle a, 1_{\bar{\rho} \otimes \rho} \rangle_{\text{End}(\iota_A)} = R_{\bar{\rho} \otimes \rho}^* \circ (1_{\bar{\rho} \otimes \rho} \otimes a) \circ R_{\bar{\rho} \otimes \rho}.$$

The same way we have a $\text{End}(\iota_B) \cong C(\Omega_B)$ -valued trace on \mathcal{B} :

$$b \in \mathcal{B}, \quad Tr_B(b) := \langle b, 1_{\rho \otimes \bar{\rho}} \rangle_{\text{End}(\iota_B)} = R_{\rho \otimes \bar{\rho}}^* \circ (1_{\rho \otimes \bar{\rho}} \otimes b) \circ R_{\rho \otimes \bar{\rho}}.$$

We will indicate for convenience $R_{\bar{\rho} \otimes \rho}^* \circ R_{\bar{\rho} \otimes \rho}$ and $R_{\rho \otimes \bar{\rho}}^* \circ R_{\rho \otimes \bar{\rho}}$ by d_A and d_B , respectively.

One defines a Fourier transform $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ as the linear map defined by

$$\mathcal{F}(a) := 1_{\rho \otimes \bar{\rho}} \otimes \bar{R}_\rho^* \circ (1_\rho \otimes a \otimes 1_{\bar{\rho}}) \circ \bar{R}_\rho \otimes 1_{\rho \otimes \bar{\rho}}.$$

and analogously $\hat{\mathcal{F}} : \mathcal{B} \rightarrow \mathcal{A}$ as

$$\hat{\mathcal{F}}(b) := 1_{\bar{\rho} \otimes \rho} \otimes R_\rho^* \circ (1_{\bar{\rho}} \otimes b \otimes 1_\rho) \circ R_\rho \otimes 1_{\bar{\rho} \otimes \rho}.$$

The maps $\mathcal{S} := \hat{\mathcal{F}} \circ \mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ and $\hat{\mathcal{S}} := \mathcal{F} \circ \hat{\mathcal{F}} : \mathcal{B} \rightarrow \mathcal{B}$ are the antipodes, and are antimultiplicative (this is an easy consequence of the conjugation relations).

PROPOSITION 4.1 *The Fourier transform preserves the inner product given by the traces Tr_A and Tr_B in the following sense:*

$$\begin{aligned} \forall a, a' \in \mathcal{A}, \quad Tr_A(a'^* \circ a) &= \theta_{\bar{\rho}}(Tr_B(\mathcal{F}(a')^* \circ \mathcal{F}(a))), \\ \forall b, b' \in \mathcal{B}, \quad Tr_B(b'^* \circ b) &= \theta_{\rho}(Tr_A(\hat{\mathcal{F}}(b')^* \circ \hat{\mathcal{F}}(b))). \end{aligned}$$

Proof. We give only a sketch of the proof and skip the tedious exposition of all the equalities.

We have $Tr_A(\hat{\mathcal{F}}(b')^* \circ \hat{\mathcal{F}}(b)) = R_{\rho}^* \circ (1_{\bar{\rho}} \otimes X) \circ R_{\rho}$, where the expression for $X \in \text{End}(\rho)$ is

$$\begin{aligned} &(\bar{R}_{\rho}^* \otimes 1_{\rho}) \circ (1_{\rho} \otimes R_{\rho}^* \otimes 1_{\bar{\rho}} \otimes 1_{\rho}) \circ (1_{\rho} \otimes 1_{\bar{\rho}} \otimes b'^* \otimes 1_{\rho}) \circ \\ &(1_{\rho \otimes \bar{\rho} \otimes \rho} \otimes (R_{\rho} \circ R_{\rho}^*)) \circ (1_{\rho} \otimes 1_{\bar{\rho}} \otimes b \otimes 1_{\rho}) \circ (1_{\rho} \otimes R_{\rho} \otimes 1_{\bar{\rho}} \otimes 1_{\rho}) \circ (\bar{R}_{\rho} \otimes 1_{\rho}) \end{aligned}$$

As R_{ρ}, \bar{R}_{ρ} is standard, we have $R_{\rho}^* \circ (1_{\bar{\rho}} \otimes X) \circ R_{\rho} = \theta_{\rho}^{-1}(\bar{R}_{\rho}^* \circ (X \otimes 1_{\bar{\rho}}) \circ \bar{R}_{\rho})$. One checks that $\bar{R}_{\rho}^* \circ (X \otimes 1_{\bar{\rho}}) \circ \bar{R}_{\rho} = Tr_B(b'^* \circ b)$, which proves the second statement. The first statement is proved analogously. \blacksquare

We also have the following proposition, which is a consequence of standardness. We omit the proof, cf. e.g. [18].

PROPOSITION 4.2 *The following relations hold:*

$$\mathcal{S} \circ \mathcal{S} = id_{\mathcal{A}} ; \quad \hat{\mathcal{S}} \circ \hat{\mathcal{S}} = id_{\mathcal{B}}$$

where by $id_{\mathcal{A}}$ and $id_{\mathcal{B}}$ we indicate the identity endomorphisms of \mathcal{A} and \mathcal{B} respectively.

We can define “convolution” products on \mathcal{A} and \mathcal{B} the following way:

$$a, a' \in \mathcal{A}, \quad a \star a' := \mathcal{F}^{-1}(\mathcal{F}(a)\mathcal{F}(a')) ; \quad b, b' \in \mathcal{B}, \quad b \star b' := \hat{\mathcal{F}}^{-1}(\hat{\mathcal{F}}(b)\hat{\mathcal{F}}(b')).$$

We restrict our attention for a moment to the case $\text{End}(\iota_A) \cong \mathbb{C}$, $\text{End}(\iota_B) \cong \mathbb{C}$. Then \mathcal{A} and \mathcal{B} are finite dimensional algebras and we are able to define a bilinear pairing between \mathcal{A} and \mathcal{B} , i.e. a non-degenerate linear form $\langle \cdot, \cdot \rangle : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{B} \rightarrow \mathbb{C}$, by $\langle a, b \rangle := d_{\rho}^{-1} Tr_A(a \mathcal{F}^{-1}(b))$, thus establishing a duality (as linear spaces) between \mathcal{A} and \mathcal{B} . This duality enables us to define coproducts $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, $\hat{\Delta} : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ by

$$\langle \Delta(a), x \otimes y \rangle := \langle a, xy \rangle, \quad a \in \mathcal{A}, x, y \in \mathcal{B}, \quad (4.1)$$

$$\langle a \otimes b, \hat{\Delta}(x) \rangle := \langle ab, x \rangle, \quad a, b \in \mathcal{A}, x \in \mathcal{B}. \quad (4.2)$$

Coassociativity of Δ and $\hat{\Delta}$ are implied by associativity of the multiplications m and \hat{m} of \mathcal{A} and \mathcal{B} respectively. We can also define counits

$$\varepsilon(a) := \langle a, 1 \rangle, \quad a \in \mathcal{A}; \quad \hat{\varepsilon}(b) := \langle 1, b \rangle, \quad b \in \mathcal{B}. \quad (4.3)$$

These operations endow the algebras \mathcal{A} and \mathcal{B} with the structure of Hopf algebras. We quote the following lemma and the following proposition from [18]:

LEMMA 4.3 (cf. [18], Lemma 6.18) *The maps*

$$\Phi_1 : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{B} \rightarrow \mathcal{D}, \quad a \otimes b \mapsto 1_{\rho} \otimes a \circ b \otimes 1_{\rho},$$

$$\Phi_2 : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{B} \rightarrow \mathcal{D}, \quad a \otimes b \mapsto b \otimes 1_{\rho} \circ 1_{\rho} \otimes a$$

are bijections.

PROPOSITION 4.4 (cf. [18], Proposition 6.19) *Let $\varepsilon, \hat{\varepsilon}, \Delta, \hat{\Delta}$ be defined as above, Then*

- $\varepsilon, \hat{\varepsilon}$ are multiplicative,
- $\Delta, \hat{\Delta}$ are multiplicative,
- S, \hat{S} are coinverses, i.e. $m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\varepsilon$, etc. (where η is the unit map : $\mathbb{C} \ni c \mapsto c1_{\mathcal{A}} \in \mathcal{A}$).
- \mathcal{A} and \mathcal{B} are finite dimensional Hopf algebras in duality, and \mathcal{C} is the Weyl algebra (in the sense of [19]) of \mathcal{A} .

The fact that the 2- C^* -category \mathcal{C} is of depth two means that there is only one isomorphism class of irreducible 1-arrows connecting A to B , the one determined by ρ . The same holds for $\bar{\rho}$. This implies that for any sub1-arrow X of $\bar{\rho} \otimes \rho$ we have $\rho \otimes X \cong \oplus_{i=1, \dots, n} \rho$, the direct sum of n copies of ρ . As the dimension is additive, we have $d_{\rho \otimes X} = n d_{\rho}$. But the dimension is multiplicative as well, i.e. $d_{\rho \otimes X} = d_X d_{\rho}$. This implies that the dimension d_X of any sub 1-arrow X of $\bar{\rho} \otimes \rho$ is an integer. Such elements $\{X \text{ sub1-arrow of } \bar{\rho} \otimes \rho\}$ form a rational tensor category. In fact, as $\bar{\rho} \otimes \rho \otimes \bar{\rho} \otimes \rho \cong \oplus_{1, \dots, d_{\bar{\rho} \otimes \rho}} \bar{\rho} \otimes \rho$, all isomorphism classes of 1-arrows in \mathcal{C} connecting A to A appear in this set. One can construct a faithful tensor functor from this category into the category of finite Hilbert spaces assigning to each X the complex Hilbert space of dimension d_X . The natural transformations of this functor have the structure of a Hopf algebra, and one can show that this is exactly the Hopf algebra \mathcal{A} introduced above (see for example [18], Proposition 6.20).

Now let's return to the general case. We fix an $\omega \in \Omega_A$. Then the 1-arrows $\theta_{\rho}^{*-1}(\omega) \xleftarrow{\rho} \omega$, $\omega \xleftarrow{\bar{\rho}} \theta_{\rho}^{*-1}(\omega)$ in the category $\mathcal{C}^{\omega, A}$ satisfy all the conditions of the above propositions. In particular $\text{End}(\rho \otimes \bar{\rho})_{\theta_{\rho}^{*-1}(\omega)} = \mathcal{B}_{\theta_{\rho}^{*-1}(\omega)}$ and $\text{End}(\bar{\rho} \otimes \rho)_{\omega} = \mathcal{A}_{\omega}$ are finite dimensional Hopf algebras in duality.

Depth two of $\theta_{\rho}^{*-1}(\omega) \xleftarrow{\rho} \omega$ in the category $\mathcal{C}^{\omega, A}$ implies that the associated dimension $d_{\mathcal{A}|\omega}$ has integer values and its square coincides with the dimension (as a vector spaces) of the algebra $\mathcal{A}|\omega$. The same conclusion applies to $\mathcal{B}_{\theta_{\rho}^{*-1}(\omega)}$. As this function is continuous with respect to ω , we conclude that $d_{\mathcal{A}}$ is a

constant function on Ω_A . Thus the dimensions (as vector spaces) of the fibre algebras $\mathcal{B}_{\theta_\rho^{*-1}(\omega)}$ and \mathcal{A}_ω are constant respect to ω . Lemma 3.16 tells us that for each $\omega \in \Omega$ we can find a neighbourhood U and an algebraic embedding of $U \times \mathcal{A}|_\omega$ into $\mathcal{A}|_U$. This embedding is actually surjective, as the fibre algebras of $\mathcal{A}|_U$ have all the same finite dimension. Thus all fibre algebras \mathcal{A}_ω are isomorphic. i.e. \mathcal{A} is a locally trivial bundle. The same conclusion applies to \mathcal{B} .

Thus we have the following: \mathcal{A} and \mathcal{B} are locally trivial C^* -algebra bundles over Ω_A and Ω_B , with fibres isomorphic to finite dimensional algebras \mathcal{A}^0 and \mathcal{B}^0 respectively.

We can view \mathcal{B} as a $C(\Omega_A)$ -valued Hilbert module by means of the isomorphism θ_ρ . Thus for $f \in C(\Omega_A)$, we have $\theta_\rho(f)b = b\theta_\rho(f) \in \mathcal{B}$, $\forall b \in \mathcal{B}$. The $C(\Omega_A)$ -valued inner product is given by $\theta_\rho^{-1}(\langle \cdot, \cdot \rangle_{\text{End}(\iota_B)})$ and we can form the tensor product $\mathcal{A} \otimes_{C(\Omega_A)} \mathcal{B}$ where, for example, $a \otimes_{C(\Omega_A)} \theta_\rho(f)b = af \otimes_{C(\Omega_A)} b$, for any $a \in A$, $b \in B$, $f \in C(\Omega_A)$. In other words the usual tensor product of the fibre bundles relative to \mathcal{A} and \mathcal{B} . Analogously, we can define a $C(\Omega_A)$ -valued linear non-degenerate form

$$\langle \cdot, \cdot \rangle : \mathcal{A} \otimes_{C(\Omega_A)} \mathcal{B} \rightarrow C(\Omega_A), \quad \langle a, b \rangle := d_\rho^{-1} \text{Tr}_{\mathcal{A}}(a\mathcal{F}^{-1}(b)).$$

This form is well defined, as one checks from the definition of \mathcal{F}^{-1} that $f \otimes \mathcal{F}^{-1}(b) = \mathcal{F}^{-1}(\theta_\rho(f) \otimes b)$, $\forall f \in C(\Omega_A)$, $\forall b \in \mathcal{B}$ holds.

If we would like to think of \mathcal{A}, \mathcal{B} as continuous bundles of Hopf algebras, the natural candidate for a continuous comultiplication Δ would be defined by

$$a \in \mathcal{A}, \quad x, y \in \mathcal{B}, \quad \Delta : a \mapsto \Delta(a) \in \mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A},$$

such that

$$\langle \Delta(a), x \otimes_{C(\Omega_A)} y \rangle = \langle a, xy \rangle.$$

As mentioned above, for each point $\omega \in \Omega_A$ we can consider the category $\mathcal{C}^{\omega, A}$. The evaluation of the bilinear form $\langle \cdot, \cdot \rangle$ in ω gives the pairing between the Hopf algebras $\mathcal{A}|_\omega$ and $\mathcal{B}_{\theta_\rho^{*-1}(\omega)}$:

$$b_{\theta_\rho^{*-1}(\omega)} \in \mathcal{B}_{\theta_\rho^{*-1}(\omega)}, \quad a_\omega \in \mathcal{A}_\omega, \quad \langle a_\omega, b_{\theta_\rho^{*-1}(\omega)} \rangle := \langle a, b \rangle|_\omega$$

where $a \in A$ is such that $a|_\omega = a_\omega$, i.e. a is a continuous section in the fibre bundle associated to the (bi-)module \mathcal{A} whose value corresponding to the base point ω is the element a_ω of the finite dimensional fibre algebra \mathcal{A}^0 , and the same way $b \in B$ such that $b|_{\theta_\rho^{*-1}(\omega)} = b_{\theta_\rho^{*-1}(\omega)}$. Thus the above defined $\Delta(a)$ indeed defines a section in the fibre bundle associated to the bimodule $\mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A}$. What we have to prove is that this section is continuous, i.e. it belongs to $\mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A}$. More precisely, we will show the following

LEMMA 4.5 *Let $a \in \mathcal{A}$. Then $\forall \epsilon > 0$ there exist a finite set of $a_1^k, a_2^k \in \mathcal{A}$ such that*

$$|(\sum_k \langle a_1^k, b \rangle \langle a_2^k, c \rangle - \langle a, bc \rangle)|_\omega| < \epsilon, \quad \forall \omega \in \Omega_A, \quad \forall b, c \in \mathcal{B}, \quad \|b\| \leq 1, \|c\| \leq 1.$$

Proof. For our convenience we will choose for \mathcal{A} and \mathcal{B} the norms given by the inner product, i.e. the $C(\Omega_A)$ (resp. $C(\Omega_B)$)-valued traces $Tr_{\mathcal{A}}$ (resp. $Tr_{\mathcal{B}}$). We choose locally trivial algebraic maps $\Phi_i : \mathcal{A}|_{U'_i} \rightarrow U'_i \times \mathcal{A}^0$, $\Psi_i : \mathcal{B}|_{V'_i} \rightarrow V'_i \times \mathcal{B}^0$, where $\{U'_i\}$ and $\{V'_i\}$ are open coverings of Ω_A and Ω_B . Without loss of generality, we suppose that $V'_i = \theta_\rho^{*-1}(U'_i)$. For each $x \in \mathcal{A}^0$ we can consider the corresponding constant section in $U'_i \times \mathcal{A}^0$, which we will indicate with the same symbol. Then $\Phi_i(x)^{-1}$ will be an element in $\mathcal{A}|_{U'_i}$. The same way, for $y \in \mathcal{B}^0$, $\Psi_i^{-1}(y)$ will be an element of $\mathcal{B}|_{V'_i}$.

Then we have the following linear form on $\mathcal{A}^0 \otimes_{\mathbb{C}} \mathcal{B}^0$ defined as

$$\langle \Phi_i^{-1}(x), \Psi_i^{-1}(y) \rangle|_{\omega}, \quad \omega \in U'_i.$$

This linear form depends continuously on ω . We define

$$\Delta_i^\omega : \mathcal{A}^0 \rightarrow \mathcal{A}^0 \otimes \mathcal{A}^0, \quad x \in \mathcal{A}^0, \quad \Delta_i^\omega(x) := \sum_l x_1^l \otimes x_2^l$$

such that for any pair $y, z \in \mathcal{B}^0$

$$\left(\sum_l \langle \Phi_i^{-1}(x_1^l), \Psi_i^{-1}(y) \rangle \langle \Phi_i^{-1}(x_2^l), \Psi_i^{-1}(z) \rangle \right)|_{\omega} = \langle \Phi_i^{-1}(x), \Psi_i^{-1}(yz) \rangle|_{\omega}.$$

As \mathcal{A}^0 is finite dimensional, all norms give equivalent topologies. For our convenience we will choose the following as norm:

$$\|x\| := \sup_{\omega \in U'_i} (Tr_{\mathcal{A}} \Phi_i^{-1}(x^*x))|_{\omega}^{\frac{1}{2}}, \quad x \in \mathcal{A}^0, \quad \forall U'_i.$$

We may suppose that we have chosen the sets U'_i such that

$$\|\Phi_i(a)|_{\omega} - \Phi_i(a)|_{\omega'}\| < \epsilon, \quad \omega, \omega' \in U'_i \quad (4.4)$$

as $\Phi_i(a)$ is a continuous section of $U'_i \times \mathcal{A}^0$. We choose an ω_i in U'_i . Then, as the map Δ_i^ω is continuous respect to ω , we may also suppose that we have chosen a set $W_i \ni \omega_i$ such that

$$\|\Delta_i^\omega(\Phi_i(a)|_{\omega_i}) - \Delta_i^{\omega'}(\Phi_i(a)|_{\omega_i})\| < \epsilon, \quad \omega, \omega' \in W_i, \quad (4.5)$$

where $\Delta_i^\omega(\Phi_i(a)|_{\omega_i})$, $\Delta_i^{\omega'}(\Phi_i(a)|_{\omega_i})$ are elements of $\mathcal{A}^0 \otimes \mathcal{A}^0$ with the norm specified just above.

Letting ω_i vary arbitrarily in U'_i , the W_i form an open cover of Ω_A . We choose a finite refinement of the two open covers of Ω_A given by the $\{U'_i\}$ and $\{W_i\}$ such that both (4.4) and (4.5) hold. We will denote with little abuse of notation these sets by $\{U_i\}$ and by $\{\Phi_i\}$ the relative local charts.

REMARK 4.6 We note that for any $b \in \mathcal{B}$, $\|b\| = \|Tr_{\mathcal{B}}(b^*b)^{\frac{1}{2}}\|^{C(\Omega_B)} < 1$ and $a \in \mathcal{A}$, $\|a\| = \|Tr_{\mathcal{A}}(a^*a)^{\frac{1}{2}}\|^{C(\Omega_A)} < 1$ one has $|\langle a, b \rangle|_{\omega} < \|d_\rho\|$ by the definition of the bilinear form and Proposition 4.1.

Now let's consider the difference

$$\begin{aligned}
& |\langle \Delta(a) - (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega_i}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}| = \\
& |\langle \Delta(a) - ((\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega}(\Phi_i(a)|_{\omega_i}), b \otimes c)_{|\omega}| \\
& + \langle (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}| - \langle (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega_i}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}| \\
& \leq |\langle \Delta(a), b \otimes c \rangle_{|\omega}| - \langle (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}| \\
& + |\langle (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}| - \langle (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega_i}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}|
\end{aligned}$$

evaluated in $\omega \in U_i$. Notice that $\langle \Delta(a), b \otimes c \rangle_{|\omega}| = \langle a, bc \rangle_{|\omega}|$ (by definition of Δ) $= \langle \Phi_i^{-1}(\Phi_i(a)|_{\omega}), bc \rangle_{|\omega}|$. Also $\langle (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}| = \langle \Phi_i^{-1}(\Phi_i(a)|_{\omega_i}), bc \rangle_{\omega}$. It follows from (4.4) and the remark above that the first summand satisfies $|\langle \Phi_i^{-1}(\Phi_i(a)|_{\omega}), bc \rangle_{\omega} - \langle \Phi_i^{-1}(\Phi_i(a)|_{\omega_i}), bc \rangle_{\omega}| < \|d_{\rho}\|\epsilon$. Analogously, from continuity of Δ_i^{ω} with respect to ω and (4.5) the second summand is $< \|d_{\rho}\|^2\epsilon$.

Thus we have proven that

$$|\langle \Delta(a) - (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega_i}(\Phi_i(a)|_{\omega_i}), b \otimes c \rangle_{|\omega}| < (\|d_{\rho}^{-1}\| + \|d_{\rho}^{-1}\|^2)\epsilon, \quad \forall \omega \in U_i.$$

Now take a partition of unity $\{f_i\}$ subordinate to the open covering of Ω_A realized by the U_i . Then $\sum_i f_i ((\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta_i^{\omega_i}(\Phi_i(a)|_{\omega_i}))$ is a sum $\sum_k a_1^k \otimes_{C(\Omega_A)} a_2^k$ of elements of $\mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A}$ which satisfies the claim of the lemma (modulo the constant $(\|d_{\rho}\| + \|d_{\rho}\|^2)$). \blacksquare

An analogous result holds for \mathcal{B} . Thus we see that (4.1), (4.2), (4.3) make sense even in the case $\text{End}(\iota_A) \neq \mathbb{C}$. We can think of (4.3) as a $\text{End}(\iota_A)$ -valued counit (resp. $\text{End}(\iota_B)$ -valued counit), or as continuous sections of counits for the fibre algebras. Continuity is obvious as in the definition only continuous objects are involved. The only non trivial part is to check continuity of the maps $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ and $\hat{\Delta} : \mathcal{B} \otimes_{C(\Omega_B)} \mathcal{B}$, which has been established by the previous lemma. We have thus the following analogue of Proposition 4.4:

PROPOSITION 4.7 *Let $\mathcal{A}, \mathcal{B}, \varepsilon, \hat{\varepsilon}, \Delta, \hat{\Delta}$ be defined as above, then*

- \mathcal{A} and \mathcal{B} are locally trivial bundles of Hopf algebras, with fibres the finite dimensional algebra \mathcal{A}^0 and \mathcal{B}^0 respectively,
- $\varepsilon, \hat{\varepsilon}$ are multiplicative,
- $\Delta, \hat{\Delta}$ are multiplicative,
- S, \hat{S} are coinverses, i.e. $m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\varepsilon$, etc.
- for every $\omega \in \Omega_A$, $\mathcal{A}_{|\omega}$ and $\mathcal{B}_{|\theta_{\rho}^{*-1}(\omega)}$ are finite dimensional Hopf algebras in duality, and $\mathcal{C}_{|\omega}$ is the Weyl algebra of $\mathcal{A}_{|\omega}$.

Proof. All algebraic properties follow by applying Proposition 4.4 to each fibre algebra. Continuity of the maps follows from the above remarks and the above lemma. \blacksquare

REMARK 4.8 A notion of $C(X)$ Hopf algebra with a continuous field of coproducts over the topological space X was introduced in [3]. This is a more general definition dealing with fields (i.e. bundles) of infinite and not necessarily unital C^* -algebras over a locally compact space X . It is fairly easy to see that our example fits this definition as well.

5 Frobenius algebras and Q -systems

In this section we would like to make some remarks concerning Frobenius algebras and Q -systems. The notion of a Q -systems was introduced in the context of von Neumann algebras in [15] (where, in some sense, it plays the role of a formalisation of a sub-factor) and subsequently for general tensor C^* -categories the notion of abstract Q -system was introduced in [16]. The notion of Frobenius algebra in a tensor category is more general, but we will show that in the C^* -category case the two notions coincide. We will follow the exposition given in [18] where not only the C^* -case is treated but more general categories are considered and a general correspondence between Frobenius algebras and pairs of conjugate 1–arrows in 2–categories is studied.

DEFINITION 5.1 (cf. [18], Definition 3.1.) *Let \mathcal{A} be a strict tensor category. A Frobenius algebra in \mathcal{A} is a quintuple (λ, V, V', W, W') , where λ is an object in \mathcal{A} and $V : \iota \rightarrow \lambda$, $V' : \lambda \rightarrow \iota$, $W : \lambda \rightarrow \lambda^2$, $W' : \lambda^2 \rightarrow \lambda$ are morphisms satisfying the following conditions:*

$$W \otimes 1_\lambda \circ W = 1_\lambda \otimes W \circ W \quad (5.1)$$

$$W' \circ W' \otimes 1_\lambda = W' \circ 1_\lambda \otimes W' \quad (5.2)$$

$$V' \otimes 1_\lambda \circ W = 1_\lambda = 1_\lambda \otimes V' \circ W \quad (5.3)$$

$$W' \circ V \otimes 1_\lambda = 1_\lambda = W' \circ 1_\lambda \otimes V \quad (5.4)$$

$$W' \otimes 1_\lambda \circ 1_\lambda \otimes W = W \circ W' = 1_\lambda \otimes W' \circ W \otimes 1_\lambda. \quad (5.5)$$

DEFINITION 5.2 (cf. [18], Definition 3.3) *Two Frobenius algebras (λ, V, V', W, W') , $(\tilde{\lambda}, \tilde{V}, \tilde{V}', \tilde{W}, \tilde{W}')$ in the strict tensor category \mathcal{A} are isomorphic if there is an isomorphism $S : \lambda \rightarrow \tilde{\lambda}$ such that*

$$S \circ V = \tilde{V}, \quad V' = \tilde{V}' \circ S, \quad S \otimes S \circ W = \tilde{W} \circ S, \quad S \circ W' = \tilde{W}' \circ S \otimes S.$$

We recall the definition of conjugation in the case of a (not necessarily C^*) 2-category \mathcal{C} (the term “duality” is often used instead of “conjugation”):

DEFINITION 5.3 *A 2-category \mathcal{C} is said to have left (right) duals if for every 1-arrow $\rho : B \leftarrow A \in \mathcal{C}$ there is $\bar{\rho} : A \leftarrow B$ ($\bar{\rho} : A \leftarrow B$) together with 2-arrows $e_\rho \in \text{Hom}(\iota, \rho \otimes \bar{\rho})$, $d_\rho \in \text{Hom}(\bar{\rho} \otimes \rho, \iota)$ ($\varepsilon_\rho \in \text{Hom}(\iota, \bar{\rho} \otimes \rho)$, $\eta_\rho \in \text{Hom}(\rho \otimes \bar{\rho}, \iota)$) satisfying:*

$$\begin{aligned} 1_\rho \otimes d_\rho \circ e_\rho \otimes 1_\rho &= 1_\rho, \quad d_\rho \otimes 1_\rho \circ 1_\rho \otimes e_\rho = 1_\rho \\ \eta_\rho \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \varepsilon_\rho &= 1_{\bar{\rho}}, \quad 1_\rho \otimes \eta_\rho \circ \varepsilon_\rho \otimes 1_\rho = 1_\rho \end{aligned}$$

If $\bar{\rho} = \bar{\rho}$, $\bar{\rho}$ is said to be a two-sided dual, and we indicate it by $\bar{\rho}$.

We will assume in the sequel duals to be two-sided. Duals are automatically two sided in a $*$ -category. It is easy to see that the above definition of duality (i.e. conjugation) reduces to the one already introduced for the C^* -case.

LEMMA 5.4 *The object λ of a Frobenius algebra is self-conjugate.*

Proof. Set $e_\lambda := W \circ V$, $d_\lambda := V' \circ W'$. It is easy to see that they satisfy the claimed relations. ■

The following is an important example:

LEMMA 5.5 (cf. [18], Lemma 3.4) *Let $\rho : B \leftarrow A$ be a 1-arrow in a 2-category \mathcal{E} and let $\bar{\rho} : A \leftarrow B$ be a two sided dual with duality 2-morphisms $d_\rho, e_\rho, \varepsilon_\rho, \eta_\rho$. Positing $\lambda = \bar{\rho} \otimes \rho : A \leftarrow A$ there are V, V', W, W' such that (λ, V, V', W, W') is a Frobenius algebra in the tensor category $\mathcal{A} = \text{HOM}_{\mathcal{E}}(A, A)$.*

Proof. It suffices to choose

$$V := \varepsilon_\rho, \quad V' := d_\rho, \quad W := 1_{\bar{\rho}} \otimes e_\rho \otimes 1_\rho, \quad W' := 1_{\bar{\rho}} \otimes \eta_\rho \otimes 1_\rho.$$

It is not difficult to check that the Frobenius algebra relations hold. ■

In the sequel we will prove explicitly a similar result in the C^* -case. The following propositions show to which extent a generic Frobenius algebra can be realized as a couple of conjugate 1-arrows in a 2-category as in the example above.

DEFINITION 5.6 *An almost-2-category is defined as a 2-category except that we do not require the existence of a unit 1-arrow ι_σ for every object σ .*

PROPOSITION 5.7 (cf. [18], Proposition 3.8) *Let \mathcal{A} be a strict tensor category and $\lambda = (\lambda, V, V', W, W')$ a Frobenius algebra in \mathcal{A} . Then there is an almost-2-category \mathcal{E}_0 satisfying:*

- $\text{Obj } \mathcal{E}_0 = \{A, B\}$.
- *There is an isomorphism $I : \mathcal{A} \rightarrow \text{HOM}_{\mathcal{E}_0}(A, A)$ of tensor categories.*

- There are 1-arrows $\rho : B \leftarrow A$ and $\bar{\rho} : A \leftarrow B$ such that $\bar{\rho} \otimes \rho = I(\lambda)$.

If \mathcal{A} is \mathbb{K} -linear then so is \mathcal{E}_0 . Isomorphic Frobenius algebras give rise to isomorphic almost-2-categories.

THEOREM 5.8 (cf. [18], Theorem 3.11) *Let \mathcal{A} be a strict tensor category and $\lambda = (\lambda, V, V', W, W')$ a Frobenius algebra in \mathcal{A} . Assume that one of the following conditions is satisfied:*

- $W' \circ W = 1_\lambda$.
- A is $\text{End}(\iota)$ -linear and

$$W' \circ W = z_1 \otimes 1_\lambda,$$

where z_1 is an invertible element of the commutative monoid $\text{End}(\iota)$.

Then the completion $\mathcal{E} = \mathcal{E}^P$ of the \mathcal{E}_0 defined in Proposition (5.7) is a bicategory such that

- $\text{Obj}\mathcal{E} = \{A, B\}$.
- There is a fully faithful tensor functor $I : \mathcal{A} \rightarrow \text{HOM}_{\mathcal{E}}(A, A)$ such that for every $Y \in \text{HOM}_{\mathcal{E}}(A, A)$ there is $X \in \mathcal{A}$ such that Y is a retract (i.e. sub-1-arrow) of $I(X)$.
- There are 1-arrows $\rho : B \leftarrow A$ and $\bar{\rho} : A \leftarrow B$ and 2-arrows

$$e_\rho : \iota_B \rightarrow \rho \otimes \bar{\rho}, \quad \varepsilon_\rho : \iota_A \rightarrow \bar{\rho} \otimes \rho, \quad d_\rho : \bar{\rho} \otimes \rho \rightarrow \iota_A, \quad \eta_\rho : \rho \otimes \bar{\rho} \rightarrow \iota_B$$

satisfying the conjugation (i.e. duality) relations.

- We have the identity

$$I(\lambda, V, V', W, W') = (\bar{\rho} \otimes \rho, e_\rho, \eta_\rho, 1_\rho \otimes \varepsilon_\rho, 1_{\bar{\rho}}, 1_\rho \otimes d_\rho, 1_{\bar{\rho}})$$

of Frobenius algebras in $\text{HOM}_{\mathcal{E}}(A, A)$.

- If \mathcal{A} is a preadditive category, then \mathcal{E} is a preadditive 2-category.
- If \mathcal{A} has direct sums then \mathcal{E} has direct sums of 1-arrows.

Isomorphic Frobenius algebras $\lambda, \tilde{\lambda}$ give rise to isomorphic bicategories $\mathcal{E}, \tilde{\mathcal{E}}$.

REMARK 5.9 As we have seen, a generic Frobenius algebra can be realized as the product of a couple of 1-arrows in a bicategory. In order for these 1-arrows to be conjugate, in Theorem 5.8 additional hypotheses were required. With further requirements one can prove the universality of this construction.

DEFINITION 5.10 (cf. [18], Definition 3.13) Let \mathcal{A} be an $\text{End}(\iota)$ -linear category. A Frobenius algebra (λ, V, V', W, W') in \mathcal{A} is “strongly separable” iff

$$W' \circ W = z_1 \otimes 1_\rho,$$

$$V' \circ V = z_2,$$

where $z_1, z_2 \in \text{End}(\iota)$ are invertible. (λ, V, V', W, W') is said to be normalised if $z_1 = z_2$.

THEOREM 5.11 (cf. [18], Theorem 3.17) Let \mathcal{A} be $\text{End}(\iota)$ -linear and (λ, V, V', W, W') a strongly separable Frobenius algebra in \mathcal{A} . Let \mathcal{E} be as constructed in Theorem 5.8 and $\tilde{\mathcal{E}}$ be any bicategory such that:

- $\text{Obj } \tilde{\mathcal{E}} = \{A, B\}$.
- Idempotent 2-arrows in $\tilde{\mathcal{E}}$ split.
- There is a fully faithful tensor functor $\tilde{I} : \mathcal{A} \rightarrow \text{HOM}_{\tilde{\mathcal{E}}}(A, A)$ such that every object of $\text{HOM}_{\tilde{\mathcal{E}}}(A, A)$ is a retract of $\tilde{I}(X)$ for some $X \in \mathcal{A}$.
- There are mutually two-sided dual 1-arrows $\tilde{\rho} : B \leftarrow A$, $\tilde{\bar{\rho}} : A \leftarrow B$ and an isomorphism $\tilde{S} : I(\lambda) \rightarrow \tilde{\rho} \otimes \tilde{\bar{\rho}}$ between the Frobenius algebras $I(\lambda, V, V', W, W')$ and $(\tilde{\rho} \otimes \tilde{\bar{\rho}}, \tilde{e}_{\tilde{\rho}}, \dots)$ in $\text{HOM}_{\tilde{\mathcal{E}}}(A, A)$.

Then there is an equivalence $E : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ of bicategories such that there is a tensor isomorphism between the tensor functors \tilde{I} and $(E|_{\text{HOM}_{\mathcal{E}}(A, A)}) \circ I$.

We recall the notion of a Q -system:

DEFINITION 5.12 Let \mathcal{A} be a tensor $*$ -category. A Q -system in \mathcal{A} is a triple (λ, T, S) where λ is an object in \mathcal{A} and $T \in \text{Hom}(\iota, \lambda)$, $S \in \text{Hom}(\lambda, \lambda^2)$ are arrows satisfying the following relations:

$$T^* \otimes 1_\lambda \circ S = 1_\lambda = 1_\lambda \otimes T^* \circ S \quad (5.6)$$

$$S^* \circ S = 1_\lambda \quad (5.7)$$

$$S \otimes 1_\lambda \circ S = 1_\lambda \otimes S \circ S \quad (5.8)$$

$$S^* \otimes 1_\lambda \circ 1_\lambda \otimes S = S \circ S = 1_\lambda \otimes S^* \circ S \otimes 1_\lambda. \quad (5.9)$$

REMARK 5.13 It follows from the definition that λ is self conjugate and $S_l(\lambda) = S_r(\lambda)$.

REMARK 5.14 The relations above are the same as in the original definition of Q -system (cf. [16]) given for the case of a tensor C^* -category with simple unit. $\text{End}(\iota)$ -linearity was not explicitly assumed, as it holds trivially when $\text{End}(\iota) \cong \mathbb{C}$. Here we have only generalised the context. In case the category \mathcal{A} is $\text{End}(\iota)$ -linear, we will talk about an $\text{End}(\iota)$ -linear Q -system.

In [18] a Q -system was defined as a strongly separable Frobenius algebra $(\lambda, T, T^*, S, S^*)$ in a tensor $*$ -category (thus assuming $\text{End}(\iota)$ -linearity). In the C^* case this latter definition is almost equivalent with our definition of $\text{End}(\iota)$ -linear Q -system: given a strongly separable Frobenius algebra $(\lambda, T, T^*, S, S^*)$ it is sufficient to renormalise T and S by the invertible z_1 in order to turn S into an isometry. On the other hand, given a Q -system as defined above, $z_1 := T^* \circ T$ and $z_2 := T^* \circ S^* \circ S \circ T$ are positive and invertible on $S_l(\lambda)$, as the following lemma shows:

LEMMA 5.15 *Let (λ, T, S) be a Q -system in a tensor C^* -category. Then $T^* \circ S^* \circ S \circ T$ and $T^* \circ T$ are positive elements of $\text{End}(\iota)$ invertible on $S_l(\lambda) = S_r(\lambda)$.*

Proof. $S \circ T$, $T^* \circ S^*$ satisfy the conjugation relations for λ and this implies that $T^* \circ S^* \circ S \circ T$ is invertible on $S_l(\lambda)$ by Corollary 1.19. The inequality

$$\|S^* \circ S\| \quad T^* \circ T \geq T^* \circ S^* \circ S \circ T$$

implies that $T^* \circ T$ is invertible on $S_l(\lambda)$ as well. ■

Thus, bearing in mind the observations at the end of section 1, we see that it is always possible to embed an $\text{End}(\iota)$ -linear Q -system (λ, T, S) (in fact, the whole tensor C^* -category generated by its tensor powers) into a tensor C^* -category such that $S_l(\lambda) = \Omega$, the topological space associated to the centre of the new tensor category. In this category $T^* \circ S^* \circ S \circ T$ and $T^* \circ T$ are invertible elements in $\text{End}(\iota)$, i.e. (λ, T, S) is a strongly separable Frobenius algebra.

REMARK 5.16 We will in the sequel indicate by $(T^* \circ T)^{-1}_{|S_l(\lambda)}$ the positive element in $\text{End}(\iota) \otimes E_{|S_l(\lambda)}$ such that $(T^* \circ T)^{-1}_{|S_l(\lambda)} \circ T^* \circ T = E_{|S_l(\lambda)}$. Thus, in particular, $((T^* \circ T)^{-1}_{|S_l(\lambda)} \circ T^* \circ T) \otimes 1_\lambda = 1_\lambda$.

We will see now that the four relations in Definition 5.12 are not independent. Having assumed (5.6) to hold, we can choose any pair of (5.7), (5.8), (5.9) and the remaining relation will follow (up to isomorphism). That (5.6, 5.7, 5.8) imply (5.9) was proven in [16]. That (5.6, 5.7, 5.9) imply (5.8) was proven in [10]. It is not necessary to suppose $\text{End}(\iota) \cong \mathbb{C}$ and $\text{End}(\iota)$ -linearity plays no role in the proof.

PROPOSITION 5.17 *Let λ, S, T be as in Definition 5.12 in a (not necessarily $\text{End}(\iota)$ -linear) tensor C^* -category \mathcal{A} . Assume (5.6) to hold. Then we have the following implications*

- (5.7) + (5.8) \Rightarrow (5.9)
- (5.7) + (5.9) \Rightarrow (5.8)
- (5.8) + (5.9) \Rightarrow (5.7) for S', T' isomorphic to S, T .

Proof. (5.7) + (5.8) \Rightarrow (5.9). Consider the inequality

$$\begin{aligned} & (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (S \circ S^*) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) \leq \\ & \|S \circ S^*\| (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) = \\ & (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) \end{aligned}$$

(as $\|S \circ S^*\| = 1$). The difference $X := (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) - (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (S \circ S^*) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda)$ is a positive element in (λ, λ) . Checking that $\langle 1_\lambda, X \rangle_{\text{End}(\iota)}^{(\lambda, \lambda)} = 0$ one concludes that the inequality is actually an equality, as the above product is non-degenerate. We put $X' := (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) - S \circ S^*$. Using the preceding result, one checks that $X'^* \circ X' = 0$, which is a restatement of (5.9).

(5.7)+(5.9) \Rightarrow (5.8) One has:

$$(1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) = (1_\lambda \otimes S) \circ (S \circ S^*).$$

On the other hand:

$$\begin{aligned} & (1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) = (1_\lambda \otimes (S \circ S^*)) \circ (S \otimes 1_\lambda) = \\ & (1_\lambda \otimes ((S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S))) \circ (S \otimes 1_\lambda) = ((1_\lambda \otimes S^* \otimes 1_\lambda) \circ (S \otimes S) = \\ & (1_\lambda \otimes S^* \otimes 1_\lambda) \circ (S \otimes 1_\lambda \otimes 1_\lambda) \circ (1_\lambda \otimes S) = ((S \circ S^*) \otimes 1_\lambda) \circ (1_\lambda \otimes S) = \\ & (S \otimes 1_\lambda) \circ (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) = (S \otimes 1_\lambda) \circ (S \circ S^*). \end{aligned}$$

Thus $(1_\lambda \otimes S) \circ (S \circ S^*) = (S \otimes 1_\lambda) \circ (S \circ S^*)$, which is equivalent to (5.8), as S is an isometry.

(5.8)+(5.9) \Rightarrow (5.7). Consider the positive element $H := S^* \circ S \in (\lambda, \lambda)$. We show that $S \circ H = H \otimes 1_\lambda \circ S = 1_\lambda \otimes H \circ S$ holds:

$$\begin{aligned} H \otimes 1_\lambda \circ S &= (S^* \circ S) \otimes 1_\lambda \circ S = S^* \otimes 1_\lambda \circ (S \otimes 1_\lambda \circ S) = S^* \otimes 1_\lambda \circ (1_\lambda \otimes S \circ S) = \\ & (S^* \otimes 1_\lambda \circ 1_\lambda \otimes S) \circ S = (S \circ S^*) \circ S = S \circ H; \end{aligned}$$

the same way

$$\begin{aligned} H \otimes 1_\lambda \circ S &= (S^* \circ S) \otimes 1_\lambda \circ S = S^* \otimes 1_\lambda \circ (S \otimes 1_\lambda \circ S) = S^* \otimes 1_\lambda \circ (1_\lambda \otimes S \circ S) = \\ & (S^* \otimes 1_\lambda \circ 1_\lambda \otimes S) \circ S = (1_\lambda \otimes S^* \circ S \otimes 1_\lambda) \circ S = \\ & 1_\lambda \otimes S^* \circ (S \otimes 1_\lambda \circ S) = 1_\lambda \otimes S^* \circ 1_\lambda \otimes S \circ S = 1_\lambda \otimes H \circ S. \end{aligned}$$

Now we show that H is invertible. We have

$$1_\lambda = S^* \circ T \otimes 1_\lambda \circ T^* \otimes 1_\lambda \circ S \leq S^* \circ ((T^* \circ T) \otimes 1_\lambda \otimes 1_\lambda) \circ S = (T^* \circ T) \otimes H$$

where we have used (5.6) and the inequality $T \circ T^* \leq (T^* \circ T) \otimes 1_\lambda$ which follows by the considerations at the end of the proof of Lemma 1.16. As $T^* \circ T$ is invertible on $S_l(\lambda)$, we can write

$$(T^* \circ T)_{|S_l(\lambda)}^{-1} \otimes 1_\lambda \leq H.$$

Thus H is positive and greater or equal to a positive invertible element, and this implies that H is invertible as well.

We can define

$$S' := H^{-\frac{1}{2}} \otimes 1_\lambda \circ S = 1_\lambda \otimes H^{-\frac{1}{2}} \circ S = S \circ H^{-\frac{1}{2}} = H^{-\frac{1}{2}} \otimes H^{-\frac{1}{2}} \circ S \circ H^{\frac{1}{2}}$$

and $T' := H^{\frac{1}{2}} \circ T$. Then (λ, S', T') is a Q -system isomorphic to (λ, S, T) and S' is an isometry. \blacksquare

COROLLARY 5.18 *Each Frobenius algebra λ in a tensor C^* -category \mathcal{T} is equivalent to a Q -system.*

COROLLARY 5.19 *Each Frobenius algebra λ in an $\text{End}(\iota)$ -linear tensor C^* -category \mathcal{T} can be embedded in a new tensor C^* -category \mathcal{T}' such that the embedding is equivalent to a strongly separable Frobenius algebra.*

PROPOSITION 5.20 *Let \mathcal{A} be a tensor C^* -category. The conclusions of Theorem 5.8 are valid without assuming anyone of the two conditions stated in the hypothesis, i.e. :*

Let $\lambda = (\lambda, T, T^, S, S^*)$ be a Frobenius algebra in \mathcal{A} . Then the completion $\mathcal{E} = \mathcal{E}^P$ of the \mathcal{E}_0 defined in Proposition 5.7 is a bicategory such that*

- $\text{Obj}\mathcal{E} = \{A, B\}$.
- *There is a fully faithful tensor functor $I : \mathcal{A} \rightarrow \text{HOM}_{\mathcal{E}}(A, A)$ such that for every $Y \in \text{HOM}_{\mathcal{E}}(A, A)$ there is $X \in \mathcal{A}$ such that Y is a retract (i.e. sub-1-arrow) of $I(X)$.*
- *There are 1-arrows $\rho : B \leftarrow A$ and $\bar{\rho} : A \leftarrow B$ and 2-arrows*

$$e_\rho : \iota_B \rightarrow \rho \otimes \bar{\rho}, \quad \varepsilon_\rho : \iota_A \rightarrow \bar{\rho} \otimes \rho, \quad d_\rho : \bar{\rho} \otimes \rho \rightarrow \iota_A, \quad \eta_\rho : \rho \otimes \bar{\rho} \rightarrow \iota_B$$

satisfying the conjugation (i.e. duality) relations.

- *We have the identity*

$$I(\lambda, T, T^*, S, S^*) = (\bar{\rho} \otimes \rho, e_\rho, \eta_\rho, 1_\rho \otimes \varepsilon_\rho \otimes 1_{\bar{\rho}}, 1_\rho \otimes d_\rho \otimes 1_{\bar{\rho}})$$

of Frobenius algebras in $\text{HOM}_{\mathcal{E}}(A, A)$.

- \mathcal{E} is a preadditive 2-category.
- \mathcal{E} has direct sums of 1-arrows.

Isomorphic Frobenius algebras $\lambda, \tilde{\lambda}$ give rise to isomorphic bicategories $\mathcal{E}, \tilde{\mathcal{E}}$.

As a last step in the sequence of propositions dealing with the relationship between Q -systems and 2-categories, we quote the following, which shows that starting from a Q -system one can extend the $*$ operation and, when starting with a Q -system in a tensor C^* -category, actually recover a 2- C^* -category.

PROPOSITION 5.21 (cf. [18], Proposition 5.5) *Let \mathcal{A} be a tensor $*$ -category and λ a Q -system in \mathcal{A} . Then \mathcal{E}_0 has a positive $*$ -operation which extends the given one on \mathcal{A} . Let \mathcal{E}_* be the full sub-bicategory of \mathcal{E} whose 1-arrows are (X, P) , where X is an object in \mathcal{A} and where $P = P \circ P = P^*$. Then \mathcal{E}_* is equivalent to \mathcal{E} with positive involution $*$.*

The following proposition answers positively a question left open in [16], namely, whether a couple of conjugate elements $\rho, \bar{\rho}$ in a tensor C^* -category does give rise to a Frobenius algebra $(\bar{\rho} \otimes \rho, S, T)$ such that S is an isometry also in the case $\text{End}(\iota) \not\cong \mathbb{C}$.

PROPOSITION 5.22 *Let $B \xleftarrow{\rho} A$ be a 1-arrow in a 2- C^* -category \mathcal{C} and let $A \xleftarrow{\bar{\rho}} B$ be a conjugate for ρ . Let $\lambda := \bar{\rho} \otimes \rho$. Then there exist S, T such that (λ, S, T) satisfy the defining relations for a Q -system (5.6-5.9) in the tensor C^* -category $\mathcal{HOM}(A, A)$ generated by 1-arrows connecting the object A to itself.*

Proof. As ρ and $\bar{\rho}$ are conjugate, there exist R_ρ, \bar{R}_ρ satisfying the conjugation equations. Defining $T' := R_\rho$, $S' := (1_{\bar{\rho}} \otimes \bar{R}_\rho \otimes 1_\rho) \circ R_\rho$ we see that $T' \in \text{Hom}(\iota, \lambda)$ and $S' \in (\lambda, \lambda \otimes \lambda)$. (λ, S', T') satisfy the relations (5.6), (5.8), (5.9) (we leave to the reader the easy proof, which relies on the conjugation relations for $\rho, \bar{\rho}$). Defining $S := S' \circ (S^* \circ S)^{-\frac{1}{2}}$, $T := (S^* \circ S)^{\frac{1}{2}} \circ T'$, Proposition 5.17 tells us that (λ, S, T) satisfy all of (5.6-5.9). \blacksquare

REMARK 5.23 A different choice of solutions R, \bar{R} or of the conjugate $\bar{\rho}$ changes the Frobenius algebra only up to isomorphism.

EXAMPLE 5.24 As $R_\rho^* \circ R_\rho$ and $\bar{R}_\rho^* \circ \bar{R}_\rho$ are invertible on $S_l(\rho)$ and $S_r(\rho)$ respectively, the elements

$$E_\rho := R_\rho \circ (R_\rho^* \circ R_\rho)^{-1}_{|S_l(\rho)} \circ R_\rho^*, \quad \bar{E}_\rho := \bar{R}_\rho \circ (\bar{R}_\rho^* \circ \bar{R}_\rho)^{-1}_{|S_r(\rho)} \circ \bar{R}_\rho^*.$$

are easily seen to be projections (where we use the same notation as before, i.e. $(R_\rho^* \circ R_\rho)^{-1}_{|S_l(\rho)}$ is the element in $\text{End}(\iota_A) \otimes E_{S_l(\rho)}$ s.t. $(R_\rho^* \circ R_\rho)^{-1}_{|S_l(\rho)} \otimes (R_\rho^* \circ R_\rho) = E_{S_l(\rho)}$ and analogously for $(\bar{R}_\rho^* \circ \bar{R}_\rho)^{-1}_{|S_r(\rho)}$.) If we suppose ρ to be centrally

balanced and Assumption 2.14 to hold (thus we have $\text{End}(\iota)$ -linearity of the category generated by $\bar{\rho} \otimes \rho$) we can also suppose to have chosen R_ρ and \bar{R}_ρ such that $\bar{R}_\rho^* \circ \bar{R}_\rho = \theta_\rho(R_\rho^* \circ R_\rho)$ (it suffices to renormalise R_ρ and \bar{R}_ρ by tensoring with $(R_\rho^* \circ R_\rho)^{-\frac{1}{4}}_{|_{S_l(\rho)}} \otimes \theta_\rho^{-1}((\bar{R}_\rho^* \circ \bar{R}_\rho)^{\frac{1}{4}})$ and $(\bar{R}_\rho^* \circ \bar{R}_\rho)^{-\frac{1}{4}}_{|_{S_r(\rho)}} \otimes \theta_\rho(R_\rho^* \circ R_\rho)^{\frac{1}{4}}$).

Then the following relations hold (cf. [16]):

$$1_\rho \otimes (R_\rho \circ R_\rho^*) \circ (\bar{R}_\rho \circ \bar{R}_\rho^*) \otimes 1_\rho \circ 1_\rho \otimes (R_\rho^* \circ R_\rho) = 1_\rho \otimes (R_\rho^* \circ R_\rho) = \theta_\rho(R_\rho^* \circ R_\rho) \otimes 1_\rho.$$

This implies

$$1_\rho \otimes E_\rho \circ \bar{E}_\rho \otimes 1_\rho \circ 1_\rho \otimes E_\rho = \theta_\rho(R_\rho^* \circ R_\rho)^{-2}_{|_{S_r(\rho)}} \otimes 1_\rho \otimes E_\rho = (\bar{R}_\rho^* \circ \bar{R}_\rho)^{-2}_{|_{S_r(\rho)}} \otimes 1_\rho \otimes E_\rho.$$

Analogously one obtains

$$1_{\bar{\rho}} \otimes \bar{E}_\rho \circ E_\rho \otimes 1_\rho \circ 1_{\bar{\rho}} \otimes \bar{E}_\rho = (R_\rho^* \circ R_\rho)^{-2}_{|_{S_l(\rho)}} \otimes 1_{\bar{\rho}} \otimes \bar{E}_\rho.$$

These are the Jones relations which in the case $\text{End}(\iota) = \mathbb{C}$ lead to a representation of the Temperley-Lieb algebra related to the parameter $R_\rho^* \circ R_\rho$. The difference here is that $R_\rho^* \circ R_\rho$ is, in general, a positive function in the C^* -algebra $\text{End}(\iota) \cong C(\Omega)$. Obviously if the function $R_\rho^* \circ R_\rho$ takes values in the discrete part of the spectrum of the Jones index, i.e. $(R_\rho^* \circ R_\rho)^2(\omega) \in \{4 \cos^2 \pi/k, k \in \mathbb{N}, k \geq 3\}$, then it is a locally constant function, as it has to be continuous on the connected subspaces of Ω .

LEMMA 5.25 *Let \mathcal{A} be a tensor C^* -category and $(\lambda, S, S^*, T, T^*)$ a Frobenius algebra in \mathcal{A} . Suppose that Assumption 2.14 holds in \mathcal{A} . Then the tensor C^* -category generated by λ (i.e. its tensor powers and their sub-objects) is $\text{End}(\iota)$ -linear.*

Proof. It suffices to prove $\text{End}(\iota)$ -linearity only for λ , i.e. that for any $z \in \text{End}(\iota)$, $z \otimes 1_\lambda = 1_\lambda \otimes z$. The same relation for powers of λ and sub-objects follows immediately.

Let's suppose for the moment λ to be centrally balanced. By Assumption 2.14 there exists a $w \in \text{End}(\iota)$ such that $z \otimes 1_\lambda = 1_\lambda \otimes w$. We suppose, without loss of generality, $w, z \in E_{S_l(\lambda)} \otimes \text{End}(\iota)$. Thus we have:

$$(1_\lambda \otimes T^*) \circ (1_\lambda \otimes z \otimes 1_\lambda) \circ S = (1_\lambda \otimes T^*) \circ (1_\lambda \otimes 1_\lambda \otimes w) \circ S.$$

For the left hand side the following hold:

$$\begin{aligned} (1_\lambda \otimes T^*) \circ (1_\lambda \otimes z \otimes 1_\lambda) \circ S &= (1_\lambda \otimes z \otimes T^*) \circ S = (1_\lambda \otimes (z \circ T^*)) \circ S = \\ &= (1_\lambda \otimes z) \circ (1_\lambda \otimes T^*) \circ S = 1_\lambda \otimes z. \end{aligned}$$

For the right hand side we have:

$$(1_\lambda \otimes T^*) \circ (1_\lambda \otimes 1_\lambda \otimes w) \circ S = ((1_\lambda \otimes T^*) \circ S) \otimes w = 1_\lambda \otimes w.$$

Thus $1_\lambda \otimes z = 1_\lambda \otimes w$, i.e. $z = w$.

Now consider $\lambda = \oplus_i \lambda_i$, where each λ_i is centrally balanced. Then for any z in $\text{End}(\iota)$ we have $z \otimes 1_\lambda = z \otimes (\oplus_i 1_{\lambda_i}) = \oplus_i (1_{\lambda_i} \otimes \theta_{\lambda_i}(z))$. But we have just seen that each θ_{λ_i} is trivial, i.e. $\theta_{\lambda_i}(z) = z$ for any z in $\text{Hom}(\iota, \iota)$.

Thus $z \otimes 1_\lambda = 1_\lambda \otimes z$. ■

COROLLARY 5.26 *Let \mathcal{A} be a tensor C^* -category for which Assumption 2.14 holds. Then each Frobenius algebra $\lambda \in \mathcal{A}$ is an $\text{End}(\iota)$ -linear Q -system in the tensor C^* -category generated by the tensor powers of λ itself.*

6 Conclusions

In all of the present work a fundamental role has been played by the C^* -property of the norm and the conjugation relations. A bundle structure of the spaces of 2-arrows appears, whereas in preceding works it was (implicitly or explicitly) a starting hypothesis. The fact that the bundle structure is preserved by the \circ composition (Proposition 2.3) is a direct consequence of the conjugation equations. In order to give a reasonable description of the behaviour of this structure under the \otimes composition, we have introduced an additional hypothesis, i.e. Assumption 2.14. This has enabled us to describe our initial 2- C^* -category as a collection of 2- C^* -categories with simple units indexed by the elements of a compact topological space (the categories $C^{\omega_0 \mathcal{A}}$ at the end of Section 2), a structure resembling that of fibre bundles which assumes a particularly nice form in the case of tensor categories (Proposition 2.32). This result suggests that tensor C^* -categories with non-simple units might play an important role for a duality theory of compact groupoids, as the simple unit case has played for compact groups ([5]). Properties analogous to the case of simple units have been proven (as, for example, the finite dimensionality of the fibres), as well continuity of the sections in this “fibre picture”. Assumption 2.14 seems general enough to handle many interesting cases. It would be interesting to study to which extent one can consider this hypothesis as valid, or give some counterexamples.

A second open question is that of the existence of standard solutions. We gave some partial answers, i.e. a “weak” positive answer (Proposition 3.18) in the general case and a “global” positive answer in the case of locally trivial bundles (Proposition 3.20). We don’t know the answer for the general case (nor do we have any counterexamples).

The bundle approach seems to have proven itself fruitful giving an example of bundles of Hopf algebras in Section 4.

Finally, the remarks in Section 5 show that, once more, the conjugation relations together with the C^* -property of the norm may hide more structure than what appears at first glance.

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